Analyse de quelques problèmes liés à l'équation de Ginzburg-Landau

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Introduction

Premier chapitre

Etude d'un problème de bifurcation associé à une fonction convexe, asymptotiquement linéaire

On étudie le problème

$$(1) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{dans } \Omega \\ \quad u = 0 & \text{sur } \partial\Omega \end{cases}$$

où:

- Ω est un ouvert borné connexe régulier de $\mathbf{R}^N$;

- $f : \mathbf{R} \to \mathbf{R}$ est une application de classe C^1, convexe, non négative, telle que $f(0) > 0$ et $f'(0) > 0$;

- λ est un paramètre positif.

On sait (voir, par exemple, [BN]), qu'il existe $\lambda^\star \in (0, \infty)$ tel que

- il y a une solution de (1) pour chaque $\lambda < \lambda^\star$;

- si $\lambda > \lambda^\star$, il n'y a aucune solution;

- pour $\lambda < \lambda^\star$ il existe une solution minimale $u(\lambda)$. De plus, $u(\lambda)$ est l'unique solution stable du problème (1) et l'application $\lambda \longmapsto u(\lambda)$ est convexe et croissante.

Quelques questions naturelles concernant l'étude du problème (1) sont:

i) l'existence d'une solution si $\lambda = \lambda^*$;

ii) le comportement des solutions $u(\lambda)$ pour $\lambda \nearrow \lambda^\star$;

iii) l'existence et le comportement d'autres solutions.

Dans le cas où f est sur-linéaire et sous-critique, Crandall et Rabinowitz ont démontré (voir [CR]) qu'il existe $u_\star = \lim\limits_{\lambda \nearrow \lambda^\star} u(\lambda)$ dans $C^1(\overline{\Omega})$. Dans le cas où f est sur-critique, la géometrie de Ω devient significative.

En collaboration avec P. Mironescu, on a étudié le cas où f est asymptotiquement linéaire, c'est-à-dire

$$(2) \qquad \lim_{t \to \infty} \frac{f(t)}{t} = a \in (0, \infty)\,.$$

Le comportement des solutions $u(\lambda)$ pour $\lambda \nearrow \lambda^\star$ varie selon la position du graphe de f par rapport la droite $y = ax$. L'étude du comportement asymptotique de $u(\lambda)$ est liée à l'observation que $u(\lambda)$ est positive et sur-harmonique. Donc, d'après un théorème classique, il y a deux posibilités quand $\lambda \nearrow \lambda^\star$:

i) $u(\lambda)$ converge uniformément (à une sous-suite près) vers $+\infty$ sur tout compact de Ω.

ii) $u(\lambda)$ converge uniformément (à une sous-suite près) dans $L^1_{\mathrm{loc}}(\Omega)$.

Soit λ_1 la première valeur propre de $-\Delta$ dans $H^1_0(\Omega)$. Les résultats qu'on a obtenus sont les suivants:

Théorème 1. *Si $f(t) \geq at$ pour tout t, alors*
i) $\lambda^* = \dfrac{\lambda_1}{a}$.
ii) $\lim_{\lambda \to \lambda^*} u(\lambda) = \infty$, *uniformément sur les sous-ensembles compacts de* Ω.
iii) $u(\lambda)$ *est l'unique solution de (1)+(2) pour* $\lambda \in (0, \lambda^*)$.
iv) le problème (1)+(2) n'a pas de solution si $\lambda = \lambda^*$.

Théorème 2. *S'il existe $t_0 \in \mathbf{R}$ tel que $f(t_0) < at_0$, alors*
i) $\lambda^* \in (\dfrac{\lambda_1}{a}, \dfrac{\lambda_1}{\lambda_0})$, *où* $\lambda_0 = \min_{t>0} \dfrac{f(t)}{t}$.
ii) le problème (1)+(2) admet une seule solution, u^*, *pour* $\lambda = \lambda^*$.
iii) $\lim_{\lambda \to \lambda^*} u(\lambda) = u^*$, *uniformément sur* Ω.
iv) si $\lambda \in (0, \dfrac{\lambda_1}{a}]$, $u(\lambda)$ *est l'unique solution du problème (1)+(2).*
v) si $\lambda \in (\dfrac{\lambda_1}{a}, \lambda^*)$, *le problème (1)+(2) a au moins une solution instable* $v(\lambda)$.
De plus, pour tout choix de $v(\lambda)$ *on a*
vi) $\lim_{\lambda \searrow \frac{\lambda_1}{a}} v(\lambda) = \infty$, *uniformément sur les sous-ensembles compacts de* Ω.
vii) $\lim_{\lambda \nearrow \lambda^*} v(\lambda) = u^*$, *uniformément sur* Ω.

L'existence d'une solution instable $v(\lambda)$ est prouvée en appliquant le théorème du col d'Ambrosetti-Rabinowitz à une fonctionnelle perturbée. On donne aussi quelques estimations sur la vitesse de croissance de $u(\lambda)$ vers $+\infty$ dans les conditions du Théorème 1.

Deuxième chapitre

L'équation de Ginzburg-Landau

L'étude du comportement asymptotique de l'équation de Ginzburg-Landau a été initiée dans une série de travaux par F. Bethuel, H. Brezis et F. Hélein (voir [BBH1-4]) et H. Brezis, F. Merle et T. Rivière (voir [BMR1-2]). Il s'agit de l'étude des points critiques de la fonctionnelle de Ginzburg-Landau

$$E_\varepsilon(u) = \frac{1}{2}\int_G \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2}\int_G (1- \mid u \mid^2)^2 \tag{3}$$

dans la classe

$$H^1_g(G) = \{u \in H^1(G,);\ u = g \text{ sur } \partial G\}$$

ainsi que leur comportement asymptotique quand $\varepsilon \to 0$. Ici, G est un domaine borné et régulier de $\mathbf{R}^2$ et $g \in C^\infty(\partial G, S^1)$. Les points critiques de E_ε vérifient l'équation de Ginzburg-Landau

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2}\, u_\varepsilon(1- \mid u_\varepsilon \mid^2)w & \text{dans } G \\ u_\varepsilon = g & \text{sur } \partial G\,. \end{cases} \tag{4}$$

Première partie

Solutions périodiques de l'équation $-\Delta v = v(1- \mid v \mid^2)$ dans $\mathbf{R}$ et $\mathbf{R}^2$

Un changement d'échelle permet d'étudier le problème (4) dans le domaine $\dfrac{G}{\varepsilon}$. Donc, le comportement asymptotique pour $\varepsilon \to 0$ des solutions de l'équation de Ginzburg-Landau nous amène à l'étude des solutions du problème

$$-\Delta v = v(1- \mid v \mid^2) \quad \text{dans } \mathbf{R}^2\,. \tag{5}$$

On étudie (avec P. Mironescu) les solutions périodiques de l'équation de Ginzburg-Landau en dimensions 1 et 2. Dans la première partie, pour $T > 0$ fixé, on cherche les solutions $v : \mathbf{R} \to$ de

$$-v'' = v(1- \mid v \mid^2) \quad \text{dans } \mathbf{R}^2 \tag{6}$$

et ayant T comme période principale. Pour chacune de ces solutions et pour $x_0 \in \mathbf{R}$, $\alpha \in$ avec $\mid \alpha \mid = 1$, l'application

$$x \longmapsto \alpha v(x_0 \pm x) \tag{7}$$

est aussi une solution. Pour éliminer cette situation, on établit d'abord pour les solutions de (6) une forme canonique:

$$\begin{cases} v_1(0) = a > 0 \\ v_1'(0) = 0 \\ v_2(0) = 0 \\ v_2'(0) = b \geq 0 \quad , \end{cases} \tag{8}$$

où $v = v_1 + iv_2$ et $a = \max |v|$. Le système (6)+(8) donne toutes les solutions de (6) qui sont distinctes du point de vue géométrique, c'est-à-dire qui ne peuvent pas être obtenues l'une de l'autre par un procédé du type (7).

Le résultat principal est

Théorème 1. *i) Si $T \leq 2\pi$, il n'y a aucune solution T-périodique.*

ii) Si $T > 2\pi$, il existe une unique solution réelle (c'est-à-dire, avec $v_2 \equiv 0$) de (6)+(8).

iii) Il existe $T_1 > 2\pi$ tel que, pour chaque $2\pi < T \leq T_1$, toutes les solutions T-périodiques de (6)+(8) sont la solution réelle de ii), ainsi que

$$v(x) = \sqrt{1 - \frac{4\pi^2}{T^2}}\, e^{i\frac{2\pi}{T}x} \,, \quad \textit{pour chaque } x \in \mathbf{R}\,.$$

iv) Pour chaque $T > T_1$, il y a d'autres solutions T-périodiques que celles trouvées à iii).

v) Pour chaque $T > 0$, le nombre de solutions T-périodiques de (6)+(8) est fini.

vi) Une borne inférieure pour le nombre de solutions $T-$périodiques est donnée par

$$\frac{5}{8}T^2 + O(T\,\log T) \quad \textit{quand } T \to \infty\,.$$

Dans $\mathbf{R}^2$, les résultats qu'on a obtenus dépendent essentiellement du parallélogramme P des périodes. On démontre que si P est suffisamment petit, alors il n'existe aucune solution non-constante de (5). Si P est un rectangle suffisament grand, alors il existe des solutions P-périodiques réelles du problème (5).

Deuxième partie

Sur l'équation de Ginzburg-Landau avec poids

On suppose que la donnée au bord g a un degré topologique $d = \deg(g, \partial G) > 0$. On considère un poids $w \in C^1(\overline{G}, \mathbf{R})$, $w > 0$ dans $\overline{G}$ et on se propose d'étudier l'énergie de Ginzburg-Landau correspondante:

$$E_\varepsilon^w(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (1-|u|^2)^2 w\ .$$

Soit u_ε un minimiseur de E_ε^w dans la classe $H_g^1(G, \mathbf{R}^2)$. F. Bethuel, H. Brezis et F. Hélein (voir [BBH2], [BBH4]) ont étudié le comportement des minimiseurs et la configuration limite dans le cas $w \equiv 1$ et ont introduit la notion d'énergie renormalisée.

Dans [4] et [5] on a étudié (avec C. Lefter) les mêmes problèmes pour le cas d'un poids régulier et positif, en donnant ainsi une reponse au problème ouvert No. 2 de [BBH4], p. 137. On démontre essentiellement que le comportement des minimiseurs est du même type que dans le cas $w \equiv 1$, la seule difference apparaîssant dans l'expression de l'énergie renormalisée et, donc, dans la localisation des singularités à la limite. Notre résultat est le suivant:

Théorème 1. *Il existe une suite $\varepsilon_n \to 0$ et exactement d points $a_1, \cdots, a_d$ dans G tels que*

$$u_{\varepsilon_n} \to u_\star \quad \text{dans } H^1_{\mathrm{loc}}(\overline{G} \setminus \{a_1, ..., a_d\}; \mathbf{R}^2)\,,$$

où $u_\star$ est l'application harmonique canonique associée aux singularités $a_1, \cdots, a_d$ de degrés +1 et à la donnée au bord g.
De plus, si $W(b)$ signifie l'énergie renormalisée associée à la configuration $b = (b_1, \cdots, b_d)$ de degrés $\overline{d} = (+1, \cdots, +1)$, alors $a = (a_1, \cdots, a_d)$ minimise la fonctionnelle

$$\widetilde{W}(b) = W(b) + \frac{\pi}{2}\sum_{j=1}^{d} \log w(b_j)$$

parmi toutes les configurations $b = (b_1, \cdots, b_d)$ de d points distincts dans G.

On a

$$\lim_{n\to\infty} \{E^w_{\varepsilon_n}(u_{\varepsilon_n}) - \pi d \mid \log \varepsilon_n \mid\} = W(a) + \frac{\pi}{2}\sum_{j=1}^{d} \log w(a_j) + d\gamma\,,$$

où γ est une constante universelle.

Un autre résultat qui caracterise le comportement asymptotique des minimiseurs est

Théorème 2. *Soit*

$$W_n = \frac{1}{4\varepsilon_n^2}(1-|u_{\varepsilon_n}|^2)^2 w\,.$$

Alors la suite (W_n) converge dans la topologie faible $\star$ de $C(\overline{G})$ vers

$$W_\star = \frac{\pi}{2} \sum_{j=1}^{d} \delta_{a_j} \ .$$

L'expression de l'énergie renormalisée $\widetilde{W}$ permet, en utilisant les résultats de Bethuel, Brezis et Hélein concernant la valeur de la différentielle de W, de prouver une propriété du type "vanishing gradient" pour le cas d'un tel poids. Soit Φ_0 l'unique solution du problème

$$\begin{cases} \Delta\Phi_0 = 2\pi \displaystyle\sum_{j=1}^{k} d_j \delta_{b_j} \ , & \text{dans } G \\ \dfrac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau \ , & \text{sur } \partial G \\ \displaystyle\int_{\partial G} \Phi_0 = 0 & \end{cases}$$

et, pour chaque $j = 1, \cdots, d$,

$$S_j(x) = \Phi_0(x) - \log \mid x - b_j \mid$$

$$R_0(x) = S_j(x) - \sum_{i \neq j} \log \mid x - b_i \mid \ .$$

Notre résultat est le suivant:

Théorème 3. *Les propriétés suivantes sont équivalentes:*
i) $a = (a_1, ..., a_d)$ est un point critique de l'énergie renormalisée $\widetilde{W}$.
ii) $\nabla S_j(a_j) = \dfrac{1}{4} \dfrac{\nabla w(a_j)}{w(a_j)}$, pour chaque j.
iii) $\nabla H_j(a_j) = \dfrac{1}{4w(a_j)} \left(-\dfrac{\partial w}{\partial x_2}(a_j), \dfrac{\partial w}{\partial x_1}(a_j) \right)$, pour chaque j.
iv) $\nabla R_0(a_j) + \displaystyle\sum_{i \neq j} \dfrac{a_j - a_i}{\mid a_j - a_i \mid^2} = \dfrac{1}{4} \dfrac{\nabla w(a_j)}{w(a_j)}$, pour chaque j.

Comme dans [BBH4], Chapitre I.4, on peut définir l'énergie renormalisée en considérant un problème variationnel dans un domaine avec des trous. Avec la méthode "shrinking holes" de Bethuel, Brezis et Hélein on démontre

Théorème 4. *Soit*

$$\widetilde{W}(b, \overline{d}, g) = W(b, \overline{d}, g) + \frac{\pi}{2} \left(\sum_{j=1}^{k} d_j^2 \log w(b_j) \right),$$

où $W(b,\overline{d},g)$ représente l'énergie renormalisée associée à la configuration $b=(b_1,\cdots,b_k)$ de degrés $\overline{d}=(d_1,\cdots,d_k)$ et à la donnée au bord g. Pour $\eta>0$ suffisamment petit, soit u_η un minimiseur de E_ε^w dans

$$G_\eta^w = G \setminus \bigcup_{j=1}^{k} \overline{B}\left(b_j, \frac{\eta}{\sqrt{w(b_j)}}\right).$$

Alors

$$\frac{1}{2}\int_{G_\eta^w} \mid \nabla u_\eta \mid^2 = \pi(\sum_{j=1}^{k} d_j^2) \mid \log\eta \mid + \widetilde{W}(b,\overline{d},g) + O(\eta), \quad \text{quand } \eta \to 0\,.$$

Ce résultat montre que l'énergie renormalisée $\widetilde{W}$ représente ce qu'il reste de l'énergie après qu'on enlève l'énergie "du noyau" $\pi d \mid \log\eta \mid$.

Troisième partie

Comportement asymptotique des minimiseurs de l'énergie de Ginzburg-Landau avec un poids qui s'annule

On continue dans [6] (en collaboration avec C. Lefter) l'étude des minimiseurs de l'énergie de Ginzburg-Landau, cette fois-ci pour un poids qui s'annule.

Soit $x_0 \in G$ et $w \in C^1(\overline{G},\mathbf{R})$ tels que $w(x_0)=0$, $w>0$ dans $\overline{G}\setminus\{x_0\}$ et $w(x) \sim \mid x-x_0 \mid^p$ dans un voisinage de x_0, où $p>0$. Notre résultat sur la convergence des minimiseurs u_ε de E_ε^w est le suivant:

Théorème 1. *Pour chaque suite $\varepsilon_n \to 0$, il existe une sous-suite (designée aussi par ε_n), k points $a_1,\cdots,a_k$ dans G et des entiers strictement positifs $d_0,d_1,\cdots,d_k$ avec $d_1+\cdots d_k=d$ tels que (u_{ε_n}) converge dans $H^1_{\mathrm{loc}}(\overline{G}\setminus\{x_0,a_1,\cdots,a_k\};\mathbf{R}^2)$ vers $u_\star$, qui est l'application harmonique canonique à valeurs dans S^1 associée aux points $x_0,a_1,\cdots,a_k$ avec les degrés correspondants positifs $d_0,d_1,\cdots,d_k$ et à la donnée au bord g.*

Le nombre de points qui s'accrochent à la limite vers le zéro du poids dépend de l'ordre de croissance $p>0$ de w autour de x_0. Plus précisément, soit $w(x) = \mid x-x_0 \mid^p + f(\mid x \mid)\cdot \mid x-x_0 \mid^{p+1}$ dans un petit voisinage de x_0, où $f:\mathbf{R}\to\mathbf{R}$ est une application de classe C^1. On démontre

Théorème 2. *1) Soit $p>0$ un nombre réel qui n'est pas un entier multiple de 4. Alors*

i) Si $d \le \frac{p}{4}+1$, alors $d_0=d$.

ii) Si $d > \frac{p}{4} + 1$, *alors* $d_0 = \left[\frac{p}{4}\right] + 1$, *où* $[x]$ *désigne la partie entière du nombre réel* x. *De plus, la configuration limite* $a = (x_0, a_1, \cdots, a_k)$ *avec les degrés correspondants* $\overline{d} = (d_0, +1, \cdots, +1)$ *minimise l'énergie renormalisée*

$$\widehat{W}(b) = W(b, \overline{d}, g) + \frac{\pi}{2}\, p \sum_{j=1}^{k} \log \mid b_j \mid, \qquad b = (x_0, b_1, \cdots, b_k)\,.$$

2) Soit p *un entier multiple de 4. Si* $d < \frac{p}{4} + 1$, *alors* $d_0 = d$. *Le cas où* p *est un entier multiple de 4 est un cas critique, au sens que si* $d \geq \frac{p}{4} + 1$, *alors* d_0 *peut avoir différentes valeurs. Par exemple, si* $G = B_1$, $x_0 = 0$ *et* $w(x) =\mid x \mid^p$, *on a le même résultat que dans le cas 1).*

On donne un exemple pour $G = B_1$, $x_0 = 0$, $d = \frac{p}{4} + 1$, $w(x) =\mid x \mid^p$ dans un voisinage de x_0, mais $d_0 = \frac{p}{4}$, donc $k = 1$.

Quatrième partie

Problèmes de minimisation et les énergies renormalisées correspondantes

En collaboration avec C. Lefter on étudie dans [7] quelques problèmes de minimisation liés à l'énergie de Ginzburg-Landau.

1) *Singularités et degrés prescrits.*

Soit $a = (a_1, \cdots, a_k)$ une configuration de points distincts dans G et $\overline{d} = (d_1, \cdots, d_k) \in {}^k$. Soit $\deg\,(g, \partial G) = d = d_1 + \cdots + d_k$. Pour $\rho > 0$ suffisamment petit, soit

$$\Omega_\rho = G \setminus \bigcup_{i=1}^{k} \overline{B(a_i, \rho)}\ ,\ \ \Omega = G \setminus \{a_1, ...a_k\}\ .$$

Soit v_ρ un minimiseur de l'énergie $\int_{\Omega_\rho} \mid \nabla v \mid^2$ dans la classe

$$\mathcal{F}_\rho = \{v \in H^1(\Omega_\rho; S^1);\ \deg(v, \partial G) = d \text{ et } \deg(v, \partial B(a_i, \rho)) = d_i,\ \text{pour } i = 1, ..., k\}\,.$$

Théorème 1. *On a l'estimation asymptotique*

$$\frac{1}{2}\int_{\Omega_\rho} \mid \nabla v_\rho \mid^2 = \pi\left(\sum_{i=1}^{k} d_i^2\right) \log \frac{1}{\rho} + \widetilde{W}(a, \overline{d}) + O(\rho)\ , \quad \textit{quand } \rho \to 0.$$

De plus, l'énergie renormalisée $\widetilde{W}(a,\overline{d})$ est liée à l'énergie renormalisée $W(a,\overline{d},g)$ définie dans [BBH4] par la formule

$$\widetilde{W}(a,\overline{d}) = \inf_{\substack{g:\partial G\to S^1\\ \deg(g,\partial G)=d}} W(a,\overline{d},g)$$

et l'infimum est atteint.

Pour le cas $G = B_1$ et $g(\theta) = e^{di\theta}$ on trouve des formules explicites pour les deux énergies renormalisées. Plus précisément, on démontre

Théorème 2. *On a*

$$W(a,\overline{d},g) == -\pi\sum_{i\neq j} d_i d_j \,\log\mid a_i - a_j\mid -\pi\sum_{i,j} d_i d_j \,\log\mid 1 - a_i\overline{a}_j\mid \ .$$

$$\widetilde{W}(a,\overline{d}) = -\pi\sum_{i\neq j} d_i d_j \,\log\mid a_i - a_j\mid +\pi\sum_{i,j} d_i d_j \,\log\mid 1 - a_i\overline{a}_j\mid \ .$$

2) *Une restriction supplémentaire pour la classe des fonctions test.*

Pour $A > 0$ fixé, soit w_ρ un minimiseur de $\displaystyle\int_{\Omega_\rho} \mid\nabla v\mid^2$ dans la classe

$$\mathcal{F}_{\rho,A} = \{v\in\mathcal{F}_\rho\,;\int_{\partial G}\mid\frac{\partial v}{\partial\tau}\mid^2\leq A\}\ .$$

Notre résultat est

Théorème 3. *On a*

$$\frac{1}{2}\int_{\Omega_\rho}\mid\nabla w_\rho\mid^2 = \pi\left(\sum_{j=1}^k d_j^2\right)\log\frac{1}{\rho} + \widetilde{W}_A(a,\overline{d}) + o(1)\ ,\quad \textit{quand } \rho\to 0\,.$$

De plus, l'énergie renormalisée $\widetilde{W}_A(a,\overline{d})$ est liée à $W(a,\overline{d},g)$ par

$$\widetilde{W}_A(a,\overline{d}) = \inf\{W(a,\overline{d},g);\ \deg(g;\partial G) = d\ \text{ et }\ \int_{\partial G}\mid\frac{\partial g}{\partial\tau}\mid^2\leq A\}\ .$$

3) *Une classe de minimiseurs de l'énergie de Ginzburg-Landau.*

Au lieu de considérer les minimiseurs de E_ε lorsqu'on prescrit la donnée au bord (comme dans [BBH4]), on est tenté de minimiser l'énergie de Ginzburg-Landau pour de degré au bord préscrit et le module 1 des fonctions test sur ∂G. Mais l'infimum de E_ε dans cette classe de fonctions n'est pas atteint, comme ont observé F. Bethuel, H. Brezis et F. Hélein. Donc, il est naturel de considérer, pour $A > 0$ fixé, les minimiseurs u_ε de E_ε dans la classe

$$\mathcal{H}_{d,A} = \{u \in H^1(G;\mathbf{R}^2);\ \mid u \mid = 1 \ \text{ sur } \ \partial G,\ \deg(u,\partial G) = d \ \text{ et } \ \int_{\partial G} \mid \frac{\partial u}{\partial \tau} \mid^2 \leq A\}\ .$$

On démontre

Théorème 4. *Pour chaque suite $\varepsilon_n \to 0$ il existe une sous-suite (désignée aussi par ε_n) et exactement d points $a_1, \cdots, a_d$ dans G tels que*

$$u_{\varepsilon_n} \to u_\star \quad \text{dans } H^1_{\text{loc}}(\overline{G} \setminus \{a_1, \cdots, a_d\};\mathbf{R}^2)\ ,$$

où $u_\star$ est l'application harmonique canonique à valeurs dans S^1 et singularités $a_1, \cdots, a_d$ de degrés +1. De plus, la configuration $a = (a_1, \cdots, a_d)$ minimise la fonctionnelle

$$\widetilde{W}_A(a,\overline{d}) := \min\{W(a,\overline{d},g);\ \deg(g;\partial G) = d \ \text{ et } \ \int_{\partial G} \mid \frac{\partial g}{\partial \tau} \mid^2 \leq A\}\ .$$

Cinquième partie

L'énergie renormalisée associée à une application harmonique

Soit $G \subset \mathbf{R}^2$ un domaine borné, régulier et simplement connexe et $g \in C^1(\partial G, S^1)$ telle que $\deg(g,\partial G) = d > 0$. Etant donné une configuration $a = (a_1, \cdots, a_k)$ de points distincts dans G et $\overline{d} = (d_1, \cdots, d_k) \in {}^k$ tel que $d_1 + \cdots d_k = d$, F. Bethuel, H. Brezis et F. Hélein ont introduit dans [BBH4] la notion d'application harmonique canonique $u_0 : \Omega = G \setminus \{a_1, \cdots, a_k\} \to S^1$ associée à $(a, \overline{d}, g)$ comme

$$u_0(z) = \left(\frac{z - a_1}{\mid z - a_1 \mid}\right)^{d_1} \cdots \left(\frac{z - a_k}{\mid z - a_k \mid}\right)^{d_k} \cdot e^{i\varphi_0(z)} \qquad \text{si } \ z \in G,$$

où

$$\begin{cases} \Delta\varphi_0 = 0 & \text{dans } \ G \\ \ \ u_0 = g & \text{sur } \ \partial G\ . \end{cases}$$

Toute application harmonique $u : \Omega \to S^1$ avec $u = g$ sur ∂G et $\deg(u, a_j) = d_j$ pour $j = 1, \cdots, k$ a la forme

(9) $$u = e^{i\psi} u_0 \quad \text{dans } \Omega\,,$$

où

$$
(10) \qquad \begin{cases} \psi(x) = \sum_{j=1}^{k} c_j \log \mid x - a_j \mid + \phi(x) \\ \psi = 0 \quad \text{sur } \partial G \\ \Delta\phi = 0 \quad \text{dans } G. \end{cases} .
$$

On introduit dans [8] (avec C. Lefter) une notion d'énergie renormalisée associée à une application harmonique u. Cette notion coïncide avec l'énergie renormalisée définie par Bethuel, Brezis et Hélein dans [BBH4] si $u = u_0$. Notre résultat est

Théorème 1. *Pour chaque application harmonique de la forme (9),*

$$
\lim_{p \nearrow 2} \left\{ \frac{1}{2} \int_G \mid \nabla u \mid^p - \frac{\pi}{2-p} \sum_{j=1}^{k} (c_j^2 + d_j^2) \right\} + \frac{\pi}{2} \sum_{j=1}^{k} (c_j^2 + d_j^2) \cdot \log \left(\sum_{j=1}^{k} (c_j^2 + d_j^2) \right) =: W(u)
$$

existe et est fini. De plus,

$$
W(u) = \lim_{\rho \to 0} \left\{ \frac{1}{2} \int_{G_\rho} \mid \nabla u \mid^2 - \pi \left(\sum_{j=1}^{k} (c_j^2 + d_j^2) \right) \log \frac{1}{\rho} \right\} .
$$

En utilisant cette evaluation asymptotique on trouve une formule explicite pour l'énergie renormalisée $W(u)$. On démontre

Théorème 2. *Pour chaque application harmonique u,*

$$
W(u) = W(a, \overline{d}, g) - \pi \sum_{j=1}^{k} c_j \phi_j(a_j) =
$$

$$
= W(u_0) - \pi \sum_{i \neq j} c_i c_j \log \mid a_i - a_j \mid - \pi \sum_{j=1}^{k} c_j \phi(a_j) \ ,
$$

où ϕ a été défini dans (10).

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A bifurcation problem associated to a convex, asymptotically linear function

Petru MIRONESCU and Vicenţiu D. RĂDULESCU

Abstract -We consider the bifurcation problem associated to a convex, asymptotically linear function and we study the behaviour of the stable solution and the existence and related properties of the unstable solutions.

Un problème de bifurcation associé à une fonction
convexe, asymptotiquement linéaire

Résumé - *On considère le problème de bifurcation associé à une fonction convexe, asymptotiquement linéaire et on étudie le comportement des solutions stable et instables, ainsi que l'existence de ces dernières.*

Version française abrégée - Dans cette note on considère le problème

$$\begin{cases} -\Delta u = \lambda f(u) & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \end{cases} \tag{1}$$

dans les conditions suivantes : Ω est un ouvert borné connexe régulier de $\mathbf{R}^N$; $f : \mathbf{R} \to \mathbf{R}$ est une fonction de classe C^1, convexe, non négative, telle que $f(0) > 0$ et $f'(0) > 0$. De plus, f est une fonction asymptotiquement linéaire vers ∞, c'est-à-dire,

$$\lim_{t\to\infty} \frac{f(t)}{t} = \lim_{t\to\infty} f'(t) = a \in (0, +\infty)$$

On suppose que λ est un paramètre positif et on cherche u dans $C^2(\Omega) \cap C(\overline{\Omega})$.

Sous ces hypothèses, on sait (voir [1]) qu'il existe $\lambda^* \in (0, \infty)$ tel que pour tout $\lambda < \lambda^*$ (resp. $\lambda > \lambda^*$), le problème (1) admet une solution (n'a aucune solution). Enfin, pour $\lambda < \lambda^*$, il existe une solution minimale $u(\lambda)$. De plus, $u(\lambda)$ est une solution stable et l'application $\lambda \mapsto u(\lambda)$ est convexe et croissante.

On se propose d'étudier les questions suivantes :

i) l'existence de plusieures solutions;

ii) l'existence d'une solution pour $\lambda = \lambda^*$;

iii) le comportement de la deuxième solution.

Dans ce cadre, nos résultats principaux sont les suivants:

THÉORÈME 1. - *Si $f(t) \geq at$ pour tout t, alors* *i)* $\lambda^* = \frac{\lambda_1}{a}$.

ii) $\lim_{\lambda \to \lambda^*} u(\lambda) = \infty$, *uniformément sur les sous-ensembles compacts de* Ω.

iii) $u(\lambda)$ *est l'unique solution de (1) pour* $\lambda \in (0, \lambda^*)$.

iv) (1) n'a pas de solutions si $\lambda = \lambda^*$.

THÉORÈME 2. - S'il existe $t_0 \in \mathbf{R}$ *tel que* $f(t_0) < at_0$, *alors*

i) $\lambda^* \in (\frac{\lambda_1}{a}, \frac{\lambda_1}{\lambda_0})$, *où* $\lambda_0 = \min_{t>0} \dfrac{f(t)}{t}$.

ii) (1) admet une seule solution, u^*, *pour* $\lambda = \lambda^*$.

iii) $\lim_{\lambda \to \lambda^*} u(\lambda) = u^*$, *uniformément sur* Ω.

iv) Si $\lambda \in (0, \frac{\lambda_1}{a}]$, $u(\lambda)$ *est l'unique solution de (1).*

v) Si $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$, *le problème (1) a au moins une solution instable* $v(\lambda)$.

De plus, pour tout choix de $v(\lambda)$ *on a*

vi) $\lim_{\lambda \to \frac{\lambda_1}{a}} v(\lambda) = \infty$, *uniformément sur les sous-ensembles compacts de* Ω.

vii) $\lim_{\lambda \to \lambda^*} v(\lambda) = u^*$, *uniformément sur* Ω.

On utilise les notations suivantes: si $\alpha \in L^\infty(\Omega)$, alors $\lambda_j(-\Delta - \alpha)$ et $\varphi_j(-\Delta - \alpha)$ sont la j-ème valeur propre (resp. la j-ème fonction propre) de l'opérateur $-\Delta - \alpha$. Si $\alpha = 0$, on les note λ_j et φ_j. On suppose toujours que $\varphi_1 > 0$ et que $\|\varphi_j\|_{L^2(\Omega)} = 1$.

Pour la démonstration de ces deux résultats un argument essentiel est le Lemme 3, qui montre que $u(\lambda)$ vérifie l' alternative suivante: ou bien $u(\lambda)$ converge vers ∞ uniformément sur les compacts de Ω, ou bien $u(\lambda)$ converge vers une solution du problème (1). L'existence d'une solution instable est obtenue via le théorème de Ambrosetti-Rabinowitz.

INTRODUCTION - We study the problem

$$(1) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where: Ω is a smooth bounded connected open set in $\mathbf{R}^N$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$, λ is a positive parameter, $f \in C^1(\mathbf{R}, \mathbf{R})$ is convex, nonnegative, with $f(0) > 0$ and $f'(0) > 0$. Moreover, we suppose that f is asymptotically linear in the sense that

$$(2) \qquad \lim_{t \to \infty} \frac{f(t)}{t} = a \in (0, \infty)$$

Under these hypotheses it is known (see [1]) that there exists $\lambda^* \in (0, \infty)$ such that

i) (1) has no solution for $\lambda > \lambda^*$.

ii) (1) has solution for every $\lambda \in (0, \lambda^*)$.

iii) when $\lambda \in (0, \lambda^*)$ there exists a minimal solution, $u(\lambda)$, which can also be described as the unique solution u such that

$$\lambda_1(-\Delta - \lambda f'(u)) > 0 \tag{3}$$

(Such solutions are called *stable*).

iv) $u(\lambda)$ increases with λ.

Here and in what follows, if $\alpha \in L^\infty(\Omega)$, then $\lambda_j(-\Delta - \alpha)$ and $\varphi_j(-\Delta - \alpha)$ denote the j-th eigenvalue (eigenfunction, respectively) of $-\Delta - \alpha$. We always suppose $\varphi_1 > 0$ and $\int_\Omega \varphi_j^2 = 1$. If $\alpha = 0$ we write λ_j and φ_j.

In this paper we are concerned with the following questions:

i) what happens when $\lambda = \lambda^*$,

ii) the behaviour of $u(\lambda)$ for λ near λ^*,

iii) under what circumstances (1) has solutions different from $u(\lambda)$.

The main results are the following:

THEOREM 1.- *If $f(t) \geq at$ for each t, then:*

i) $\lambda^* = \frac{\lambda_1}{a}$

ii) $\lim_{\lambda \to \lambda^*} u(\lambda) = \infty$, *u.c.s.* Ω.

iii) $u(\lambda)$ is the only solution of (1) when $\lambda \in (0, \lambda^*)$.

iv) (1) has no solution when $\lambda = \lambda^*$.

THEOREM 2.- *If there exists t_0 such that $f(t_0) < at_0$, then:*

i) $\lambda^* \in (\frac{\lambda_1}{a}, \frac{\lambda_1}{\lambda_0})$

ii) (1) has exactly one solution, say u^*, when $\lambda = \lambda^*$.

iii) $\lim_{\lambda \to \lambda^*} u(\lambda) = u^*$ *u.*Ω.

iv) when $\lambda \in (0, \frac{\lambda_1}{a}]$, (1) has no solution but $u(\lambda)$.

v) when $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$, (1) has at least an unstable solution, say $v(\lambda)$.

Moreover, for each choice of $v(\lambda)$ we have

vi) $\lim_{\lambda \to \frac{\lambda_1}{a}} v(\lambda) = \infty$ *u.c.s.* Ω.

vii) $\lim_{\lambda \to \lambda^*} v(\lambda) = u^*$ *u.*Ω.

Here, $\lambda_0 = \min_{t>0} \dfrac{f(t)}{t}$, a solution u is called *unstable* if $\lambda_1(-\Delta - \lambda f'(u)) \leq 0$, while *u.c.s.* and *u.* mean uniformly on compact subsets (uniformly, resp.).

After the sketches of the proofs, we discuss the problem of the order of convergence to ∞ in the Theorems 1 and 2. As all integrals are taken over Ω, we shall omit this in our writing.

I.*Proofs of the Theorems 1 and 2*

We mention first some auxiliary results:

LEMMA 1. *Let* $\alpha \in L^\infty(\Omega)$, $w \in H_0^1(\Omega) \setminus \{0\}, w \geq 0$, *be such that* $\lambda_1(-\Delta - \alpha) \leq 0$ *and*

$$-\Delta w \geq \alpha w \tag{4}$$

Then $\lambda_1(-\Delta - \alpha) = 0, -\Delta w = \alpha w$ *and* $w > 0$ *in* Ω.

This follows multiplying (4) by $\varphi_1(-\Delta - \alpha)$ and integrating by parts.

LEMMA 2. *i)* $\lambda^* \geq \frac{\lambda_1}{a}$.

ii) if $f(t) = at + b, b > 0$, *then* $\lambda^* = \frac{\lambda_1}{a}$ *and (1) has no solution when* $\lambda = \lambda^*$.

iii) if (1) has solution when $\lambda = \lambda^*$, *it is necessarily unstable.*

iv) (1) has at most one solution when $\lambda = \lambda^*$.

v) $u(\lambda)$ *is the only solution of (1) such that* $\lambda_1(-\Delta - \lambda f'(u)) \geq 0$.

Proof.- i) 0 and the solution $u \in H_0^1(\Omega)$ of $-\Delta u = \lambda(au + f(0))$ are sub and supersolution for (1) when $\lambda \in (0, \frac{\lambda_1}{a})$.

ii) It suffices to prove the second part, which follows, by contradiction, multiplying by φ_1 and integrating.

iii) Otherwise, in view of the implicit function theorem, λ^* would not be maximal.

iv) If v_1 is such a solution, then $v_2 = \lim\limits_{\lambda \to \lambda^*} u(\lambda)$ is also a solution and $v_2 \leq v_1$. With $w = v_1 - v_2 \geq 0$, we have $-\Delta w \geq f'(v_2)w$. Hence, either $w = 0$ or $w > 0$, but then $f(v_1) = av_1 + f(0)$. The last possibility contradicts ii).

v) As in *iv)*, if v were such a solution different from $u(\lambda)$, then $f'(v) = f'(u(\lambda))$.

LEMMA 3. *The following assertions are equivalent:*

i) $\lambda^* = \frac{\lambda_1}{a}$.

ii) (1) has no solution in $C^2(\Omega) \cap C(\overline{\Omega})$ *when* $\lambda = \lambda^*$.

iii) $\lim\limits_{\lambda \to \lambda^*} u(\lambda) = \infty$ *u.c.s.* Ω.

Proof.- i)$\Longrightarrow$*ii)* Any such solution u is a priori unstable. But this forces f to be linear in $[0, \max\limits_\Omega u]$, which contradicts Lemma 2.

ii)$\Longrightarrow$*iii)* It is enough to show that $u(\lambda)$ is bounded in $L^2(\Omega)$. Suppose the contrary. Then, by the Theorem 4.1.9., p. 94, of [2], $u(\lambda)$ converges in $L^1_{loc}(\Omega)$ to some u^*. If $u(\lambda) = k(\lambda)w(\lambda)$, $k(\lambda) > 0, \int w^2(\lambda) = 1$, we get the existence of some w, weak $\star$ cluster point of $(w(\lambda))$ in $H_0^1(\Omega)$ such that $w \geq 0$, $\int w^2 = 1$ and $-\Delta w = 0$.

Obviously, *iii)*$\Longrightarrow$*ii)*. It remains to see that [*iii)* and *ii)*]$\Longrightarrow$ *i)*. If w is obtained as above, this time it verifies $-\Delta w = \lambda^* a w$. Hence, $\lambda^* a = \lambda_1$ and $w = \varphi_1$.

COROLLARY 1. *Under the hypotheses of the Lemma 3,*

$$\lim_{\lambda\to\frac{\lambda_1}{a}} \frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda) = \varphi_1 \quad u.\overline{\Omega}$$

Via a bootstrap argument and the Theorem 8.34, p. 211 in [3], we get that $w(\lambda)$ is bounded in $C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$. We apply afterwards the Arzela-Ascoli Theorem.

Proof of the Theorem 1.- i), ii), iv). Suppose (1) has a solution u when $\lambda = \frac{\lambda_1}{a}$. Then, $-\Delta u \geq \lambda_1 u$. Lemma 1 implies $f(u) = au + f(0)$, but this contradicts Lemma 2.

iii) If u is a solution, then $\lambda_1(-\Delta - \lambda f'(u)) > \lambda_1(-\Delta - \lambda_1) = 0$.

Proof of the Theorem 2.- i) Suppose $\lambda^* = \frac{\lambda_1}{a}$. Then

$$0 = \lim_{\lambda\to\frac{\lambda_1}{a}} \int \varphi_1[(\lambda_1 - a\lambda)u(\lambda) + \lambda(au(\lambda) - f(u(\lambda)))] \geq \frac{-l\lambda_1}{a} \int \varphi_1 > 0,$$

where $l = \lim_{t\to\infty} [f(t) - at] < 0$. If we suppose $\lambda^* \geq \frac{\lambda_1}{\lambda_0}$, we obtain a similar contradiction.

ii), iii), iv) are obvious.

v) is a consequence of the Ambrosetti-Rabinowitz Theorem. Let

$$\lambda \in (\frac{\lambda_1}{a}, \lambda^*), \epsilon_0 = \frac{a\lambda - \lambda_1}{2\lambda_1}, u_0 = u(\lambda), F(t) = \lambda \int_0^t f(s)\,ds, X = H_0^1(\Omega),$$

$$J_\epsilon(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(u) + \frac{\epsilon}{2}\int |\nabla(u - u_0)|^2,\ u \in X,\ \epsilon \in [0, \epsilon_0]$$

Then it is known (see [1]) that $J_0 \in C^1(X, \mathbf{R})$ and u_0 is a local minimum for J_0. Hence, u_0 is a local strict minimum for J_ϵ, $\epsilon \in (0, \epsilon_0]$.Since $\lim_{t\to\infty} \sup_{\epsilon\in[0,\epsilon_0]} J_\epsilon(t\varphi_1) = -\infty$, there exists $v_0 \in X$ with $J_\epsilon(v_0) \leq J_\epsilon(u_0)$, for each ϵ. If

$$\wp = \{p \in C([0,1], X) : \quad p(0) = u_0,\ p(1) = v_0\}$$

and $c_\epsilon = \inf_{\wp} \max_{[0,1]} J_\epsilon \circ p$, then $c_0 \leq c_\epsilon \leq \max_{[u_0,v_0]} J_0 + \dfrac{\epsilon}{2}\displaystyle\int |\nabla(v_0 - u_0)|^2$.

The variational problem satisfies a Palais-Smale type condition, in the sense that if

$$(J_{\epsilon_n}(u_n)) \text{ is bounded} \tag{5}$$

and

$$J'_{\epsilon_n}(u_n) \longrightarrow 0 \tag{6}$$

then (u_n) contains a convergent subsequence.By standard arguments, it is enough to find a subsequence bounded in L^2. Suppose the contrary: let $u_n = k_n w_n, \int w_n^2 = 1,\ k_n > 0$, $\lim_{n\to\infty} k_n = \infty$, $\epsilon_n \to \epsilon \in [0, \epsilon_0]$. Then, by (5), if we modify f such that $\lim_{t\to-\infty} f(t) = 0$, we get (up to a subsequence) $w_n \to w$, both in weak $\star$ H_0^1 and L^2 sense, with $-(1+\epsilon)\Delta w = \lambda a w^+$. Hence $w^+ = w$, which contradicts the choice of ϵ_0. Hence there exists $(v_\epsilon)_{\epsilon\in(0,\epsilon_0]}$ precompact in $H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta v_\epsilon = \lambda f(v_\epsilon) + \epsilon(u_0 - v_\epsilon) \\ J_\epsilon(v_\epsilon) = c_\epsilon > J_\epsilon(u_0) \end{cases} \tag{7}$$

(Note that this implies $v_\epsilon \neq u_0$ and v_ϵ unstable). Let v be a limit point of v_ϵ when $\epsilon \to 0$. Then v is the desired solution. Indeed, v is unstable as limit of unstable solutions.

vi) follows immediately if we show that $(v(\lambda))$ is bounded in $L^2(\Omega)$ when λ is near λ^*. The contrary would give as in Lemma 3 that $\lambda^* = \frac{\lambda_1}{a}$.

vii) If we suppose the contrary we obtain the same contradiction as in the proof of ii)$\Longrightarrow$ iii) in Lemma 3.

Further results: 1) If $\lambda_0 \geq \frac{a\lambda_1}{\lambda_2}$ (or, more generally, if $\lambda^* \leq \frac{\lambda_2}{a}$) then $v(\lambda)$ is unique. This follows from [4], p. 838. This implies that in this case v depends C^1 on λ.

2) If $\lambda^* > \frac{\lambda_2}{a}$ then there exists $\epsilon > 0$ such that $v(\lambda)$ is unique in $(\frac{\lambda_1}{a}, \frac{\lambda_2}{a} + \epsilon)$. Indeed, for $\lambda = \frac{\lambda_2}{a}$ we have, if v is an unstable solution of (1), that $\lambda_2(-\Delta - \lambda f'(v)) \geq 0$. The equality would imply that f is linear in $[0, \max_\Omega v]$, which is contradictory. Since $\lambda_1(-\Delta - \lambda f'(v)) < 0$, we get the uniqueness when $\lambda = \frac{\lambda_2}{a}$ via the previous remark and the implicit function theorem. A routine argument shows that the uniqueness remains true in a neighborhood of $\frac{\lambda_2}{a}$.

A natural question is to estimate the speed of convergence to ∞ of $u(\lambda)$ in Theorem 1. Regarding the equality

$$\int \varphi_1[(\lambda_1 - a\lambda)u(\lambda) + \lambda(au(\lambda) - f(u(\lambda)))] = 0 \tag{8}$$

one can obtain the following results:

3) If $l = \lim_{t\to\infty}[f(t) - at] \in (0, \infty)$, then

$$\frac{a(\lambda_1 - a\lambda)}{\lambda_1 l \int \varphi_1} u(\lambda) \to \varphi_1 \quad u.\overline{\Omega} \tag{9}$$

4) If $l = 0$ then

$$(\lambda_1 - a\lambda)u(\lambda) \to 0 \quad u.\overline{\Omega}$$

In this case the answer depends heavily on f. For example:

i) if $f(t) = t + \frac{1}{t+2}$ then $u(\lambda) \sim \frac{c}{\sqrt{\lambda_1 - \lambda}}\varphi_1$;

ii) if $f(t) = t + \frac{1}{(t+1)^2}$ then $u(\lambda) \to \infty$ like no power of $(\lambda_1 - \lambda)$.

Similarly,

5) If $l \in (-\infty, 0)$ then (9) is true with $v(\lambda)$ instead of $u(\lambda)$.

6) If $l = -\infty$ then

$$(\lambda_1 - a\lambda)v(\lambda) \to \infty \quad u.c.s.\Omega$$

7) In the above statements we can allow $f'(0) = 0$ if f is strictly convex near 0.

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THE STUDY OF A BIFURCATION PROBLEM ASSOCIATED TO AN ASYMPTOTICALLY LINEAR FUNCTION

Petru MIRONESCU and Vicenţiu D. RĂDULESCU

Introduction

In this paper we consider the problem

$$(1) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where: Ω is a smooth connected bounded open set in $\mathbf{R}^N$, $f : \mathbf{R} \to \mathbf{R}$ is a C^1 convex nonnegative function such that $f(0) > 0, f'(0) > 0$ and f is asymptotically linear, that is

$$\lim_{t\to\infty} \frac{f(t)}{t} = a \in (0, +\infty)$$

In what follows we suppose that λ is a positive parameter and $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

We point out some well known facts about the problem (1) (see [5] for details):

i) there exists $\lambda^* \in (0, +\infty)$ such that (1) has (has no) solution when $\lambda \in (0, \lambda^*)$ ($\lambda \in (\lambda^*, +\infty)$, resp.).

ii) for $\lambda \in (0, \lambda^*)$, among the solutions of (1) there exists a minimal one, say $u(\lambda)$.

iii) $\lambda \longmapsto u(\lambda)$ is a C^1 convex increasing function.

iv) $u(\lambda)$ can be characterized as the only solution u of (1) such that the operator $-\Delta - \lambda f'(u)$ is coercive.

In what follows, we discuss some natural problems raised by (1):

i) what can be said when $\lambda = \lambda^*$?

ii) which is the behaviour of $u(\lambda)$ when λ approaches λ^*?

iii) are there other solutions of (1) excepting $u(\lambda)$?

iv) if so, which is their behaviour?

Before mentioning our main results, we give some definitions and notations:

i) let $\lim_{t\to\infty} (f(t) - at) = l \in [-\infty, \infty)$. We say that f obeys *the monotone case (the non-monotone case)* if $l \geq 0$ ($l < 0$, resp.).

ii) if $\alpha \in L^\infty(\Omega)$ we shall denote by $\varphi_j(\alpha)$ and $\lambda_j(\alpha)$ the jth eigenfunction (eigenvalue, resp.) of $-\Delta - \alpha$. We consider that $\int_\Omega \varphi_j(\alpha)\varphi_k(\alpha) = \delta_{jk}$ and $\varphi_1(\alpha) > 0$. If $\alpha = 0$ we shall write φ_j (λ_j, resp.).

iii) a solution u of (1) is said to be *stable* if $\lambda_1(\lambda f'(u)) > 0$ and *unstable* otherwise.

iv) $u.c.s.\Omega$ and $u.\overline{\Omega}$ will mean "uniformly on compact subsets of Ω" ("uniformly on Ω", resp.).

All the integrals considered are over Ω, so that we shall omit Ω in writing.

Now we can state the main results:

THEOREM A.- *If f obeys the monotone case, then:*
i) $\lambda^* = \frac{\lambda_1}{a}$
ii) $\lim\limits_{\lambda \to \lambda^*} u(\lambda) = \infty$, *u.c.s.* Ω.
iii) $u(\lambda)$ *is the only solution of (1) when* $\lambda \in (0, \lambda^*)$.
iv) (1) has no solution when $\lambda = \lambda^*$.

THEOREM B.- *If f obeys the non-monotone case, then:*
i) $\lambda^* \in (\frac{\lambda_1}{a}, \frac{\lambda_1}{\lambda_0})$, *where* $\lambda_0 = \min\limits_{t>0} \dfrac{f(t)}{t}$
ii) (1) has exactly one solution, say u^*, *when* $\lambda = \lambda^*$.
iii) $\lim\limits_{\lambda \to \lambda^*} u(\lambda) = u^*$ $u.\overline{\Omega}$.
iv) when $\lambda \in (0, \frac{\lambda_1}{a}]$, *(1) has no solution but* $u(\lambda)$.
v) when $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$, *(1) has at least an unstable solution, say* $v(\lambda)$.
For each choice of $v(\lambda)$ *we have*
vi) $\lim\limits_{\lambda \to \frac{\lambda_1}{a}} v(\lambda) = \infty$ *u.c.s.* $\overline{\Omega}$.
vii) $\lim\limits_{\lambda \to \lambda^*} v(\lambda) = u^*$ $u.\overline{\Omega}$.

After we establish these results, we discuss the problem of the order of convergence to ∞ in the theorems A and B.

1. Proof of Theorem A

LEMMA 1. *Let* $\alpha \in L^\infty(\Omega), w \in H_0^1(\Omega) - \{0\}, w \geq 0,$ *be such that* $\lambda_1(\alpha) \leq 0$ *and*

$$-\Delta w \geq \alpha w \tag{2}$$

Then:

i) $\lambda_1(\alpha) = 0$

ii) $-\Delta w = \alpha w$

iii) $w > 0$ *in* Ω.

Proof: If we multiply (2) by $\varphi_1(\alpha)$ and integrate by parts, we obtain

$$\int \alpha\varphi_1(\alpha)w + \lambda_1(\alpha)\int \varphi_1(\alpha)w \geq \int \alpha\varphi_1(\alpha)w$$

Now this means that $\lambda_1(\alpha) = 0$ and $-\Delta w = \alpha w$. Since $w \geq 0$ and $w \not\equiv 0$, we get $w = c\varphi_1(\alpha)$ for some $c > 0$, which concludes the proof. □

LEMMA 2. (the linear case) *If* $f(t) = at + b$ *when* $t \geq 0$, *with* $a, b > 0$, *then*

i) $\lambda^* = \frac{\lambda_1}{a}$.

ii) (1) has no solution when $\lambda = \lambda^*$.

Proof: i), ii) If $\lambda \in (0, \frac{\lambda_1}{a})$ then the problem

$$\begin{cases} -\Delta u - \lambda a u = \lambda b & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

has a unique solution in $H_0^1(\Omega)$ which is positive in view of Stampacchia maximum principle (see [5]). Now Ω smooth and $-\Delta u = \lambda a u + \lambda b \in H_0^1(\Omega)$ mean $u \in H^3(\Omega)$ and so on. We get $u \in H^\infty(\Omega)$ and therefore $u \in C^\infty(\overline{\Omega})$. We have thus exhibited a smooth solution of (1) when $\lambda \in (0, \frac{\lambda_1}{a})$

We claim that (1) has no solution if $\lambda^* = \frac{\lambda_1}{a}$. For if u were such a solution, multiplying (1) by φ_1 and integrating by parts, we get $\int \varphi_1 = 0$, which contradicts $\varphi_1 > 0$. □

LEMMA 3. *i)* $\lambda^* \geq \frac{\lambda_1}{a}$.

ii) if (1) has solution when $\lambda = \lambda^*$, *it is necessarily unstable.*

iii) (1) has at most a solution when $\lambda = \lambda^*$.

iv) $u(\lambda)$ *is the only solution of (1) such that* $\lambda_1(\lambda f'(u)) \geq 0$.

Proof: *i)* It is enough to exhibit a super and sub solution for $\lambda \in (0, \frac{\lambda_1}{a})$, that is: $\underline{U}, \overline{U} \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\underline{U} \leq \overline{U}$,

$$\begin{cases} -\Delta \overline{U} \geq \lambda f(\overline{U}) & \text{in } \Omega \\ \overline{U} \geq 0 & \text{on } \partial\Omega \end{cases}$$

and that the reversed inequalities hold for $\underline{U}$ (see [5] for the method of super and subsolutions).

Take some $b > 0$ such that $f(t) \leq at + b$ for nonnegative t. Let $\overline{U}$ be the solution of (3) with $b = f(0)$ and $\underline{U} \equiv 0$. We have $f(t) \leq at + b$ for $t > 0$ and this implies $f(\overline{U}) \leq a\overline{U} + b$ in view of the positivity of $\overline{U}$. The remaining part is trivial.

ii) Suppose that (1) with $\lambda = \lambda^*$ has a solution u^* with $\lambda_1(\lambda^* f'(u^*)) > 0$. Then by the implicit function theorem applied to

$$G : \{u \in C^{2,\frac{1}{2}}(\overline{\Omega}) : \; u = 0 \text{ on } \partial\Omega\} \times \mathbf{R} \to C^{0,\frac{1}{2}}(\overline{\Omega}), \quad G(u, \lambda) = -\Delta u - \lambda f(u)$$

it follows that (1) has solution for λ in a neighbourhood of λ^*, contradicting by this the definition of λ^*.

iii) Let u be such a solution. Then u is a supersolution for (1) when $\lambda \in (0, \lambda^*)$ and therefore $u \geq u(\lambda)$ for such λ. This shows that $u(\lambda)$ (which increases with λ) tends in $L^1(\Omega)$ sense to a limit $u^* \leq u$. Since $-\Delta u(\lambda) = \lambda f(u(\lambda))$ we get $-\Delta u^* = \lambda^* f(u^*)$. In order to conclude that u^* is a solution of (1), it is enough to prove that $u^* \in H_0^1(\Omega)$ and to deduce from this first that either $-\Delta u^* \in L^{2^*}(\Omega)$ and hence $u^* \in W^{2,2^*}(\Omega)$ when $N > 2$, or $-\Delta u^* \in L^4(\Omega)$ and hence $u^* \in C^{0,\frac{1}{2}}(\overline{\Omega})$ if $N = 1, 2$ (using theorems 8.34 and 9.15 in [7]). The first case is then concluded via a bootstrap argument, while the second one using the theorem 4.3 in [7] (here $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent).

Now we claim that $u(\lambda)$ is bounded in $H_0^1(\Omega)$. Indeed, if we multiply (1) by $u(\lambda)$ and integrate by parts we get

$$\int |\nabla u(\lambda)|^2 = \lambda \int f(u(\lambda))u(\lambda) \leq \lambda^* \int u f(u)$$

Thus, $u(\lambda) \rightharpoonup u^*$ in $H_0^1(\Omega)$ if $\lambda \to \lambda^*$. Indeed, if v is a weak-$\star$ cluster point of $u(\lambda)$ when $\lambda \to \lambda^*$, then, up to a subsequence, $u(\lambda) \to v$ *a.e.* But $u(\lambda) \to u$ *a.e.* We have hence obtained that $u^* \in H_0^1(\Omega)$. The proof will be concluded if we show that $u = u^*$. Let $w = u - u^* \geq 0$. Then

$$-\Delta w = \lambda^*(f(u) - f(u^*)) \geq \lambda^* f'(u^*) w \tag{4}$$

We also have $\lambda_1(\lambda^* f'(u^*)) \leq 0$, so that lemma 1 implies that either $w = 0$ or $w > 0$, $\lambda_1(\lambda^* f'(u^*)) = 0$ and $-\Delta w = \lambda^* f'(u^*)w$. If we take (4) into account the last equality implies that f is linear in all the intervals $[u^*(x), u(x)], x \in \Omega$. It is easy to see that this forces f to be linear in $[0, \max_\Omega u]$. Let $\alpha, \beta > 0$ be such that $f(u) = \alpha u + \beta$ and $f(u^*) = \alpha u^* + \beta$. We have

$$0 = \lambda_1(\lambda^* f'(u^*)) = \lambda_1(\lambda^* \alpha) = \lambda_1 - \lambda^* \alpha,$$

that is $\lambda^* = \frac{\lambda_1}{\alpha}$. The last conclusion contradicts Lemma 2.

iv) Suppose (1) has a solution $u \neq u(\lambda)$ with $\lambda_1(\lambda f'(u)) \geq 0$. Then $u > u(\lambda)$ by the strong maximum principle (see the theorem 3.5. in [7]). Let $w = u - u(\lambda) > 0$. Then

$$-\Delta w = \lambda(f(u) - f(u(\lambda))) \leq \lambda f'(u)w \tag{5}$$

If we multiply (5) by $\varphi = \varphi_1(\lambda f'(u))$ and integrate by parts we get

$$\lambda \int f'(u)\varphi w + \lambda_1(\lambda f'(u)) \int \varphi w \leq \lambda \int f'(u)\varphi w$$

Thus, $\lambda_1(\lambda f'(u)) = 0$ and in (5) we have equality , that is f is linear in $[0, \max_\Omega u]$. Let $\alpha, \beta > 0$ be such that $f(u) = \alpha u + \beta, f(u(\lambda)) = \alpha u(\lambda) + \beta$. Then

$$0 = \lambda_1(\lambda f'(u)) = \lambda_1(\lambda f'(u(\lambda))),$$

contradiction. □

The following result is a reformulation of the theorem 4.1.9. in [9].

LEMMA 4. *Let (u_n) be a sequence of nonnegative superharmonic functions in Ω. Then*

either

i) $\lim_{n\to\infty} u_n = \infty$ *u.c.s.* Ω

or

ii) (u_n) *contains a subsequence which converges in $L^1_{loc}(\Omega)$ to some u^*.*

LEMMA 5. *The following conditions are equivalent:*

i) $\lambda^* = \frac{\lambda_1}{a}$

ii) (1) has no solution when $\lambda = \lambda^*$

iii) $\lim_{\lambda\to\lambda^*} u(\lambda) = \infty$ *u.c.s.* Ω

Proof: i)$\Longrightarrow$ ii) Suppose the contrary. Let u be such a solution. As we have already seen, $\lambda_1(\lambda^* f'(u)) \leq 0$. But $\lambda_1(\lambda^* f'(u)) \geq \lambda_1(\lambda^* a) = 0$.

Hence $\lambda_1(\lambda^* f'(u)) = 0$, that is $f'(u) = a$. As already happened, this contradicts lemma 2.

ii) $\Longrightarrow$ iii) Suppose the contrary. We prove first that $u(\lambda)$ are uniformly bounded in $L^2(\Omega)$. Suppose again the contrary. Then, up to a subsequence, $u(\lambda) = k(\lambda)w(\lambda)$ with $k(\lambda) \to \infty$ and $\displaystyle\int w^2(\lambda) = 1$.

Suppose, using again a subsequence if necessary, that $u(\lambda) \to u^*$ in $L^1_{loc}(\Omega)$.Then $\frac{\lambda}{k(\lambda)} f(u(\lambda)) \to 0$ in $L^1_{loc}(\Omega)$, that is

$$-\Delta w(\lambda) \to 0 \quad \text{in } L^1_{loc}(\Omega) \tag{6}$$

It is easy to see that $(w(\lambda))$ is bounded in $H^1_0(\Omega)$. Indeed,

$$\int |\nabla w(\lambda)|^2 = \int -\Delta w(\lambda) w(\lambda) = \int \frac{\lambda}{k(\lambda)} f(u(\lambda)) w(\lambda) \leq$$

$$\leq \lambda^* \int (a w^2(\lambda) + \frac{f(0)}{k(\lambda)} w(\lambda)) \leq \lambda^* a + c \int w(\lambda) \leq$$

$$\leq \lambda^* a + c \int \sqrt{|\Omega|} \quad (\text{for a suitable } c > 0)$$

Let $w \in H^1_0(\Omega)$ be such that,up to a subsequence,

$$w(\lambda) \to w \;\; \text{weakly in} H^1_0(\Omega) \text{ and strongly in } L^2(\Omega) \tag{7}$$

Then, by (6), $-\Delta w = 0$, and by (7), $w \in H^1_0(\Omega)$ and $\int w^2 = 1$. We have obtained the desired contradiction. Hence $(u(\lambda))$ is bounded in $L^2(\Omega)$. As above, $u(\lambda)$ is bounded in $H^1_0(\Omega)$. Let $u \in H^1_0(\Omega)$ be such that, up to a subsequence, $u(\lambda) \to u$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. Then by (1) we get that u is a $H^1_0(\Omega)$ solution of $-\Delta u = \lambda^* f(u)$. As we have already done, we get that in fact u is a solution of (1) when $\lambda = \lambda^*$. This contradiction concludes the proof.

iii)$\Longrightarrow$ii). As we have seen, if (1) has a solution when $\lambda = \lambda^*$, it is necessarily equal to $\lim\limits_{\lambda \to \lambda^*} u(\lambda)$, which cannot happen in the given context.

[iii) and ii)]$\Longrightarrow$ i) Let $u(\lambda) = k(\lambda)w(\lambda)$ with $k(\lambda)$ and $w(\lambda)$ as above. This time $\lim\limits_{\lambda \to \lambda^*} k(\lambda) = \infty$. As above we get a uniform bound for $(w(\lambda))$ in $H^1_0(\Omega)$. Let $w \in H^1_0(\Omega)$ be such that, up to a subsequence, $w(\lambda) \to w$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. Then $-\Delta w(\lambda) \to -\Delta w$ in $\mathcal{D}'(\Omega)$ and $\frac{\lambda}{k(\lambda)} f(u(\lambda)) \to \lambda^* a w$ in $L^2(\Omega)$. (The last statement will be shown out in the proof of Lemma 9). So we obtain

$$-\Delta w = \lambda^* a w, \;\; w \in H^1_0(\Omega), \;\; w \geq 0, \;\; \int w^2 = 1$$

But this means exactly that $\lambda^* = \frac{\lambda_1}{a}$ (and $w = \varphi_1$). $\square$

LEMMA 6. *The following conditions are equivalent:*

i) $\lambda^* > \frac{\lambda_1}{a}$

ii) (1) has exactly a solution,say u^**, when* $\lambda = \lambda^*$*.*

iii) $u(\lambda)$ *is converging u.* $\overline{\Omega}$ *to some* u^* *which is the unique solution of (1) when* $\lambda = \lambda^*$*.*

Proof: We have already seen that $\lambda^* \geq \frac{\lambda_1}{a}$. This makes this lemma a reformulation of the preceding one apart the fact that the limit in iii) is u. $\overline{\Omega}$. Since we know that $u(\lambda) \to u^*$ *a.e.*, it is enough to prove that $u(\lambda)$ has a limit in $C(\overline{\Omega})$ when $\lambda \to \lambda^*$. Even less, it is enough to prove that $u(\lambda)$ is relatively compact in $C(\overline{\Omega})$. This will be done via the Arzela-Ascoli Theorem if we show that $(u(\lambda))$ is bounded in $C^{0,\frac{1}{2}}(\overline{\Omega})$. Now $0 < u(\lambda) < u^*$ implies $0 < f(u(\lambda)) < f(u^*)$, which offers a uniform bound for $-\Delta u(\lambda)$ in $L^{2N}(\Omega)$. The desired bound is now a consequence of the theorem 8.34 in [7] (see also the remark from the page 212) and of the closed graph theorem. $\square$

Proof of Theorem A:

i), ii) and *iv)* will follow together if we prove one of them. We shall prove that $\lambda^* = \frac{\lambda_1}{a}$ by showing that (1) has no solution when $\lambda = \frac{\lambda_1}{a}$. For suppose u were such a solution. Then

$$-\Delta u = \lambda f(u) \geq \lambda_1 u \tag{8}$$

If we multiply (8) by φ_1 and integrate by parts we get $\lambda f(u) = \lambda_1 u$, contradicting the fact that $f(0) > 0$.

iii) taking into account the lemma 3 iv), it is enough to prove that for $\lambda \in (0, \frac{\lambda_1}{a})$ any solution u verifies $\lambda_1(\lambda f'(u)) \geq 0$. But

$$-\Delta - \lambda f'(u) \geq -\Delta - \lambda a$$

which shows that

$$\lambda_1(\lambda f'(u)) \geq \lambda_1(\lambda a) = \lambda_1 - \lambda a > 0$$

$\square$

2.Proof of Theorem B

i) We prove first that $\lambda^* \leq \frac{\lambda_1}{\lambda_0}$. For this aim, we shall see that (1) has no solution when $\lambda = \frac{\lambda_1}{\lambda_0}$. Suppose the contrary and let u be such a solution. Then multiplying (1) by φ_1 and integrating by parts we get

$$\lambda_1 \int \varphi_1 u = \lambda \int \varphi_1 f(u) \tag{9}$$

In our case, (9) becomes

$$\lambda_1 \int \varphi_1 u = \frac{\lambda_1}{\lambda_0} \int \varphi_1 f(u) \geq \lambda_1 \int \varphi_1 u$$

which forces $f(u) = \lambda_0 u$ and, as above, this contradicts $f(0) > 0$.

The remaining part of *i), ii)* and *iii)* are equivalent in view of the lemmas 3 iii) and 6. We shall prove that $\lambda^* > \frac{\lambda_1}{a}$ supposing the contrary. Then $\lim\limits_{\lambda \to \lambda^*} u(\lambda) = \infty \ \ u.c.s.\Omega$ and $\lambda^* = \frac{\lambda_1}{a}$. If we examine (9) rewritten as

$$0 = \int \varphi_1 [\lambda_1 u(\lambda) - \lambda f(u(\lambda))] = \tag{10}$$

$$= \int \varphi_1 [(\lambda_1 - a\lambda) u(\lambda) - \lambda (f(u(\lambda)) - au(\lambda))] \geq -\lambda \int \varphi_1 [f(u(\lambda)) - au(\lambda))]$$

we see that the righthand side integrand converges monotonously to $l\varphi_1$ when $\lambda \to \lambda^*$. Here $l = \lim\limits_{t \to \infty} (f(t) - at) < 0$. Passing to the limit in (10) we obtain the contradictory inequality

$$0 \geq -l\lambda \int \varphi_1 > 0$$

We have seen that $\lambda^* \leq \frac{\lambda_1}{\lambda_0}$ and we know that (1) has solution when $\lambda = \lambda^*$. This shows that $\lambda^* < \frac{\lambda_1}{\lambda_0}$.

iv) can be proved exactly in the same way as *iii)* in the theorem A.

Since all the solutions of (1) are positive, we may modify $f(t)$ as we wish for negative t. In what follows we shall suppose, additionally, that f is increasing.

For the proof of *v)* we shall use some known results that we point out in what follows:

THE AMBROSETTI-RABINOWITZ THEOREM: *Let E be a Banach space, $J \in C^1(E, \mathbf{R})$, $u_0 \in E$. Suppose that there exist $R, \rho > 0, v_0 \in E$ such that*

$$J(u) \geq J(u_0) + \rho \quad \textit{if} \quad \|u - u_0\| = R \tag{11}$$

$$J(v_0) \leq J(u_0) \tag{12}$$

Suppose that the following condition is satisfied:

(PS) *every sequence (u_n) in E such that $(J(u_n))$ is bounded in $\mathbf{R}$ and $J'(u_n) \to 0$ in E^* is relatively compact in E.*

Let

$$\mathcal{P} = \{p \in C([0,1], E) : \ p(0) = u_0, p(1) = v_0\}$$

and

$$c = \inf_{\mathcal{P}} \max_{[0,1]} F \circ p$$

Then there exists $u \in E$ such that $J(u) = c$ and $J'(u) = 0$.

Note that $c > J(u_0)$ and that is why $u \neq u_0$ (see [5] for details).

We want to find out solutions of (1) different from $u(\lambda)$, that is critical points, others than $u(\lambda)$, of

$$J : E \to \mathbf{R}, \quad J(u) = \frac{1}{2} \int |\nabla u|^2 - \int F(u)$$

where $E = H_0^1(\Omega)$ and $F(t) = \lambda \int_0^t f(s)ds$. We take $u(\lambda)$ as u_0 for each $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$.

We have

LEMMA 7. *i) $J \in C^1(E, \mathbf{R})$*
ii) For $u, v \in E$ we have $J'(u)v = \int \nabla u \cdot \nabla v - \lambda \int f(u)v$
iii) u_0 is a local minimum for J.

The proof can be found in [5].

In order to apply the Ambrosetti-Rabinowitz Theorem we transform u_0 into a local strict minimum by modifying J. Let

$$J_\epsilon : E \to \mathbf{R}, \quad J_\epsilon(u) = J(u) + \frac{\epsilon}{2} \int |\nabla(u - u_0)|^2$$

In view of the preceding lemma we obviously have
i) $J \in C^1(E, \mathbf{R})$
ii) $J'_\epsilon(u) \cdot v = \int \nabla u \cdot \nabla v - \lambda \int f(u)v + \epsilon \int \nabla(u - u_0) \cdot \nabla v$
iii) u_0 is a local strict minimum for J_ϵ if $\epsilon > 0$ (so that (11) is verified).

We prove first the existence of a v_0 good for all ϵ near 0.

LEMMA 8. *Let $\epsilon_0 = \frac{\lambda a - \lambda_1}{2\lambda_1}$. Then there exists $v_0 \in E$ such that $J_\epsilon(v_0) < J_\epsilon(u_0)$ for $\epsilon \in [0, \epsilon_0]$.*

Proof: Note that $J_\epsilon(u)$ is bounded by $J_0(u)$ and $J_{\epsilon_0}(u)$. It suffices to prove that

$$\lim_{t \to \infty} J_{\epsilon_0}(t\varphi_1) = -\infty$$

But

$$J_\epsilon(t\varphi_1) = \frac{\lambda_1}{2} t^2 + \frac{\epsilon_0}{2} \lambda_1 t^2 - \tag{13}$$

$$-\epsilon_0\lambda_1 t\int \varphi_1 u_0 + \frac{\epsilon_0}{2}\int |\nabla u_0|^2 - \int F(t\varphi_1)$$

Let $\alpha = \frac{3a\lambda+\lambda_1}{4\lambda}$. Since $\alpha < a$, there exists $\beta \in \mathbf{R}$ such that $f(s) \geq \alpha s + \beta$ for all s, which implies that $F(s) \geq \frac{\alpha\lambda}{2}s^2 + \beta\lambda s$ when $s \geq 0$. Then (13) shows that

$$\limsup_{t\to\infty} \frac{1}{t^2} J_{\epsilon_0}(t\varphi_1) \leq \frac{\lambda_1 + \epsilon_0\lambda_1 - \lambda\alpha}{2} < 0$$

because of the choice of α. □

LEMMA 9. *The condition (PS) is satisfied uniformly in ϵ, that is if*

$$(J_{\epsilon_n}(u_n)) \text{ is bounded in } \mathbf{R}, \quad \epsilon_n \in [0, \epsilon_0] \tag{14}$$

and

$$J'_{\epsilon_n}(u_n) \to 0 \text{ in } E^* \tag{15}$$

then (u_n) is relatively compact in E.

Proof: Since any subsequence of (u_n) verifies (14) and (15), it is enough to prove that (u_n) contains a convergent subsequence. It suffices to prove that (u_n) contains a bounded subsequence in E. Indeed, suppose we have proved this. Then, up to a subsequence, $u_n \to u$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and *a.e.*, and $\epsilon_n \to \epsilon$. Now (15) gives that

$$-\Delta u_n - \lambda f(u_n) - \epsilon_n \Delta(u_n - u_0) \to 0 \quad \text{in } \mathcal{D}'(\Omega)$$

Note that $f(u_n) \to f(u)$ in $L^2(\Omega)$ because $|f(u_n) - f(u)| \leq a|u_n - u|$. This shows that

$$-(1+\epsilon_n)\Delta u_n \to \lambda f(u) - \epsilon\Delta u_0 \quad \text{in } \mathcal{D}'(\Omega),$$

that is

$$-\Delta u - \lambda f(u) - \epsilon\Delta(u - u_0) = 0 \tag{16}$$

The above equality multiplied by u gives

$$(1+\epsilon)\int |\nabla u|^2 - \lambda\int uf(u) - \epsilon\lambda\int uf(u_0) = 0 \tag{17}$$

Now (15) multiplied by (u_n) gives

$$(1+\epsilon_n)\int |\nabla u_n|^2 - \lambda\int u_n f(u_n) - \epsilon_n\lambda\int u_n f(u_0) \to 0 \tag{18}$$

in view of the boundedness of (u_n). The middle term in (18) tends to $-\lambda\int uf(u)$ and the last one to $-\epsilon\lambda\int uf(u_0)$ in view of the $L^2(\Omega)$ -convergence of u_n and $f(u_n)$. Hence, if we compare the first terms in (17) and (18) we get that $\int|\nabla u_n|^2 \to \int|\nabla u|^2$, which insures us that $u_n \to u$ in $H_0^1(\Omega)$. Actually, it is enough to prove that (u_n) is (up to a subsequence) bounded in $L^2(\Omega)$. Indeed, the $L^2(\Omega)$-boundedness of (u_n) implies the $H_0^1(\Omega)$-boundedness of (u_n) as it can be seen by examining (14).

We shall conclude the proof obtaining a contradiction from the supposition that $\|u_n\|_{L^2(\Omega)} \to \infty$. Let $u_n = k_n w_n$ with $k_n > 0, \int w_n^2 = 1$ and $k_n \to \infty$. We may suppose $\epsilon_n \to \epsilon$. Then

$$(19)\qquad 0 = \lim_{n\to\infty}\frac{J_{\epsilon_n}(u_n)}{k_n^2} = \lim_{n\to\infty}[\frac{1}{2}\int|\nabla w_n|^2 - \frac{1}{k_n^2}\int F(u_n) + \frac{\epsilon_n}{2}\int|\nabla(w_n - \frac{u_0}{k_n})|^2]$$

Now

$$\frac{\epsilon_n}{2}\int|\nabla(w_n - \frac{u_0}{k_n})|^2 = \frac{\epsilon_n}{2}\int|\nabla w_n|^2 + \frac{\epsilon_n}{2k_n^2}\int|\nabla u_0|^2 - \frac{\epsilon_n\lambda}{k_n}\int w_n f(u_0)$$

Thus (19) can be rewritten

$$\lim_{n\to\infty}[\frac{1+\epsilon_n}{2}\int|\nabla w_n|^2 - \frac{1}{k_n^2}\int F(u_n)] = 0$$

But

$$|F(u_n)| = |F(k_n w_n)| \le \frac{\lambda a}{2}k_n^2 w_n^2 + \lambda b|k_n w_n|$$

because $|f(t)| \le a|t| + b$. Here $b = f(0)$. This shows that $(\frac{1}{k_n^2}\int F(u_n))$ is bounded and this must also be true for $\|w_n\|_{H_0^1(\Omega)}$. Now let $w \in H_0^1(\Omega)$ be such that (up to a subsequence) $w_n \to w$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and *a.e.*. Note that $\int w^2 = 1$. We claim that

$$(20)\qquad -(1+\epsilon)\Delta w = \lambda a w^+$$

Indeed, (15) divided by k_n gives

$$(21)\qquad (1+\epsilon_n)\int\nabla w_n\cdot\nabla v - \lambda\int\frac{f(u_n)}{k_n}v - \frac{\epsilon_n\lambda}{k_n}\int f(u_0)v \to 0$$

for each $v \in H_0^1(\Omega)$. Now

$$(1+\epsilon_n)\int\nabla w_n\cdot\nabla v \to (1+\epsilon)\int\nabla w\cdot\nabla v$$

Hence (20) can be concluded from (21) if we show that $\frac{1}{k_n}f(u_n)$ converges (up to a subsequence) to aw^+ in $L^2(\Omega)$. Now $\frac{1}{k_n}f(u_n) = \frac{1}{k_n}f(k_n w_n)$ and it is easy to see that the required limit is equal to aw^+ in the set

$$\{x \in \Omega : \ w_n(x) \to w(x) \neq 0\}$$

If $w(x) = 0$ and $w_n(x) \to w(x)$, let $\epsilon > 0$ and n_0 be such that $|w_n(x)| < \epsilon$ for $n \geq n_0$. Then

$$\frac{f(k_n w_n)}{k_n} \leq \epsilon a + \frac{b}{k_n} \text{ for such } n,$$

that is the required limit is 0. Thus, $\frac{f(u_n)}{k_n} \to aw^+$ *a.e.* Here $b = f(0)$. Now $w_n \to w$ in $L^2(\Omega)$ and thus, up to a subsequence, w_n is dominated in $L^2(\Omega)$ (see theorem IV.9 in [4]).

Since $\frac{1}{k_n}f(u_n) \leq a|w_n| + \frac{1}{k_n}b$, it follows that $\frac{1}{k_n}f(u_n)$ is also dominated. Hence (20) is now obtained. Now (20) and the maximum principle imply $w \geq 0$ and (20) becomes

$$\begin{cases} -\Delta w = \dfrac{\lambda a}{1+\epsilon} w \\ w \geq 0 \\ \displaystyle\int w^2 = 1 \end{cases} \tag{22}$$

Thus $\frac{\lambda a}{1+\epsilon} = \lambda_1$ (and $w = \varphi_1$), which contradicts the fact that $\epsilon \in [0, \epsilon_0]$ and the choice of ϵ_0. This contradiction finishes the proof of the lemma 9. □

LEMMA 10. *c_ϵ is uniformly bounded.*

Proof: The fact that J_ϵ increases with ϵ implies $c_\epsilon \in [c_0, c_{\epsilon_0}]$. □

Now we continue the proof of the theorem B *v)*: for $\epsilon \in (0, \epsilon_0]$, let $v_\epsilon \in H_0^1(\Omega)$ be such that

$$-\Delta v_\epsilon = \frac{\lambda}{1+\epsilon} f(v_\epsilon) + \frac{\lambda \epsilon}{1+\epsilon} f(u_0) \tag{23}$$

and

$$J_\epsilon(v_\epsilon) = c_\epsilon \tag{24}$$

The relation (24) and the lemmas 9 and 10 show that there exists $v \in H_0^1(\Omega)$ such that $v_\epsilon \to v$ in $H_0^1(\Omega)$ as $\epsilon \to 0$. Now (23) implies

$$-\Delta v = \lambda f(v)$$

The last assertions to be proved are that $v \neq u_0 = u(\lambda)$ and $v \in C^2(\Omega) \cap C(\overline{\Omega})$. Note that v_ϵ is a solution of (23) different from u_0 and hence unstable, in the sense that

$$\lambda_1(\frac{\lambda}{1+\epsilon} f'(v_\epsilon)) \leq 0$$

Indeed (23) is an equation of the form

$$-\Delta u = g(u) + h(x)$$

where g is convex and positive and h is positive. Then, if it has solutions, it has a minimal one, say u, with $\lambda_1(g'(u)) \geq 0$ (see [5]). Now the proof of the lemma 3 *iv)* shows that for all other solutions v we have $\lambda_1(g'(v)) < 0$. In our case, u_0 stands for u and v_ϵ for v. All we have to prove now is that the limit of a sequence of unstable solutions is also unstable, which will be done in

LEMMA 11. *Let* $u_n \rightharpoonup u$ *in* $H_0^1(\Omega)$ *and* $\mu_n \to \mu$ *be such that* $\lambda_1(\mu_n f'(u_n)) \leq 0$. *Then* $\lambda_1(\mu f'(u)) \leq 0$.

Proof: The fact that $\lambda_1(\alpha) \leq 0$ is equivalent to the existence of a $\varphi \in H_0^1(\Omega)$ such that

$$\int |\nabla \varphi|^2 \leq \int \alpha \varphi^2 \text{ and } \int \varphi^2 = 1$$

follows from the Hilbert-Courant min-max principle.

Let $\varphi_n \in H_0^1(\Omega)$ be such that

$$\int |\nabla \varphi_n|^2 \leq \int \mu_n f'(u_n) \varphi_n^2 \tag{25}$$

and

$$\int \varphi_n^2 = 1 \tag{26}$$

Since $f' \leq a$, (25) shows that (φ_n) is bounded in $H_0^1(\Omega)$. Let $\varphi \in H_0^1(\Omega)$ be such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ in $H_0^1(\Omega)$. Then the righthand side of (25) converges, up to a subsequence, to $\mu \int f'(u)\varphi^2$. This can be seen by extracting from (φ_n) a subsequence dominated in $L^2(\Omega)$ as in the theorem IV.9 in [4]. Since

$$\int \varphi^2 = 1 \text{ and } \int |\nabla \varphi|^2 \leq \liminf \int |\nabla \varphi_n|^2,$$

we get the desired result.

The fact that $v \in C^2(\Omega) \cap C(\overline{\Omega})$ follows via a bootstrap argument:

$$v \in H_0^1(\Omega) \Rightarrow f(v) \in L^{2^*}(\Omega) \Rightarrow v \in W^{2,2^*}(\Omega) \Rightarrow ...$$

The key facts are:

a) if $v \in L^p(\Omega)$ then $f(v) \in L^p(\Omega)$

b) an elliptic regularity result (theorem 9.15 in [7]).

c) the Sobolev embeddings.

vi) Suppose the contrary. Then there are $\mu_n \to \frac{\lambda_1}{a}$, v_n an unstable solution of (1) with $\lambda = \mu_n$, and $v \in L^1_{loc}(\Omega)$ such that $v_n \to v$ in $L^1_{loc}(\Omega)$

We claim first that (v_n) cannot be bounded in $H_0^1(\Omega)$. Otherwise, let $w \in H_0^1(\Omega)$ be such that, up to a subsequence, $v_n \to w$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Then

$$-\Delta v_n \to -\Delta w \text{ in } \mathcal{D}'(\Omega) \text{ and } f(v_n) \to f(w) \text{ in } L^2(\Omega),$$

which shows that $-\Delta w = \frac{\lambda_1}{a} f(w)$.

It follows that $w \in C^2(\Omega) \cap C(\overline{\Omega})$, that is w is a solution of (1). From Lemma 11 it follows that

$$\lambda_1(\frac{\lambda_1}{a} f'(w)) \leq 0 \tag{27}$$

Now (27) shows that $w \neq u(\frac{\lambda_1}{a})$, which contradicts *iv)* of the Theorem.

The fact that (v_n) is not bounded in $H_0^1(\Omega)$ implies that (v_n) is not bounded in $L^2(\Omega)$. Indeed, we have seen that the $L^2(\Omega)$-boundedness implies the $H_0^1(\Omega)$ one. So, let $v_n = k_n w_n$, where $k_n > 0$, $\int w_n^2 = 1$ and up to a subsequence $k_n \to \infty$.

We have

$$-\Delta w_n = \frac{\mu_n}{k_n} f(u_n) \to 0 \text{ in } L^1_{loc}(\Omega)$$

(and hence we have convergence also in the distribution sense) and (w_n) is seen to be bounded in $H_0^1(\Omega)$ with an already provided argument. If w is a $\star$-cluster point of (w_n) in $H_0^1(\Omega)$, we obtain $-\Delta w = 0$ and $\int w^2 = 1$, the desired contradiction.

vii) As before, it is enough to prove the $L^2(\Omega)$-boundedness of $v(\lambda)$ near λ^* and to use the uniqueness property of u^*. Suppose the contrary. Let $\mu_n \to \lambda^*$, $\|v_n\|_{L^2(\Omega)} \to \infty$, where v_n are the corresponding solutions of (1). If we write again $v_n = k_n w_n$, then

$$-\Delta w_n = \frac{\mu_n}{k_n} f(u_n) \tag{28}$$

The fact that the righthand side of (28) is bounded in $L^2(\Omega)$ implies that (w_n) is bounded in $H_0^1(\Omega)$. Let w be such that up to a subsequence $w_n \to w$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. A computation already done shows that

$$-\Delta w = \lambda^* a w,\ w \geq 0 \text{ and } \int w^2 = 1,$$

which forces λ^* to be $\frac{\lambda_1}{a}$. This contradiction concludes the proof. □

3. Some further remarks

As we have seen in the proofs of the Theorems 1 and 2, we have that

i) in the monotone case, $\lim\limits_{\lambda \to \frac{\lambda_1}{a}} \dfrac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda) = \varphi_1$ in $H_0^1(\Omega)$.

ii) in the non-monotone case, $\lim\limits_{\lambda \to \frac{\lambda_1}{a}} \dfrac{1}{\|v(\lambda)\|_{L^2(\Omega)}} v(\lambda) = \varphi_1$ in $H_0^1(\Omega)$.

It is natural to try to find out:

i) if the above limits continue to exist in a more restrictive sense, say in $C(\overline{\Omega})$.

ii) which is the asymptotic behaviour of $\|u(\lambda)\|_{L^2(\Omega)}$ and $\|v(\lambda)\|_{L^2(\Omega)}$ when λ is near $\frac{\lambda_1}{a}$.

It is easy to answer the first question. We have

PROPOSITION 1. *i) in the monotone case,*

$$\lim_{\lambda \to \frac{\lambda_1}{a}} \frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda) = \varphi_1 \quad \textit{in } C^1(\overline{\Omega})$$

ii) in the non-monotone case,

$$\lim_{\lambda \to \frac{\lambda_1}{a}} \frac{1}{\|v(\lambda)\|_{L^2(\Omega)}} v(\lambda) = \varphi_1 \quad \textit{in } C^1(\overline{\Omega})$$

Proof: i) The proof is essentially the same as for the Lemma 6: it is enough to prove that $(\frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda))$ is relatively compact in $C^1(\overline{\Omega})$ (when λ is near $\frac{\lambda_1}{a}$), which can be done by showing that it is bounded in $C^{1,\frac{1}{2}}(\overline{\Omega})$. But this follows from the fact that the above set is bounded in $H_0^1(\Omega)$ and a bootstrap argument (note that a uniform bound for $w(\lambda) = \frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda)$ in some $L^p(\Omega),\ 1 < p < \infty$ provides a uniform bound for $-\Delta w(\lambda)$ in $L^p(\Omega)$ for the same p).

ii) is identical with *i)*. □

Moreover, we have

PROPOSITION 2. *If $w(\lambda)$ is either $\frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda)$ or $\frac{1}{\|v(\lambda)\|_{L^2(\Omega)}} v(\lambda)$, then $\frac{\varphi_1}{w(\lambda)}$ is uniformly bounded when λ is near $\frac{\lambda_1}{a}$.*

Proof: Note that the strong maximum principle implies that $\frac{\partial w(\lambda)}{\partial \nu} < 0$ on $\partial\Omega$ and hence $\frac{\varphi_1}{w(\lambda)}$ can be extended to a continuous function on $\overline{\Omega}$ by setting

$$\frac{\varphi_1}{w(\lambda)}(x) = \frac{\frac{\partial \varphi_1}{\partial \nu}(x)}{\frac{\partial w(\lambda)}{\partial \nu}(x)} \quad \text{for} \quad x \in \partial\Omega.$$

LEMMA 12. *There exists* $\epsilon_0 > 0$ *such that if*

$$\omega_0 = \{x \in \mathbf{R}^N : \ d(x, \partial\Omega) < \epsilon_0\}$$

then

i) for each $x \in \omega_0$ *there is a unique* $x_0 \in \partial\Omega$ *such that* $d(x, \partial\Omega) = |x - x_0|$.

ii) if $\Pi(x) = x_0$, *then* $\Pi \in C^1(\omega_0)$ $(x, x_0$ *are as above*).

iii) if $|x - \Pi(x)| = \epsilon$ *then* $x = \Pi(x) - \epsilon\nu(\Pi(x))$ *or* $x = \Pi(x) + \epsilon\nu(\Pi(x))$, *according to the case* $x \in \Omega$ *or* $x \notin \Omega$.

iv) if $x \in \Omega$ *then* $[x, \Pi(x)) \subset \Omega$.

The proof can be found in [10]. □

Let $\omega = \omega_0 \cap \Omega$ and $K = \Omega \setminus \omega$. Since $w(\lambda) \to \varphi_1 \quad u.\overline{\Omega}$, for λ close enough to $\frac{\lambda_1}{a}$ we have $w(\lambda)_{|K} > \dfrac{1}{2} \min\limits_K \varphi_1$, that is $\frac{\varphi_1}{w(\lambda)} < c$ in K for such λ and a suitable c. If $x \in \omega$, let $x_0 = \Pi(x)$. Then

$$\text{(29)} \qquad \frac{\varphi_1(x)}{w(\lambda, x)} = \frac{\varphi_1(x) - \varphi_1(x_0)}{w(\lambda, x) - w(\lambda, x_0)} = \frac{-\epsilon \frac{\partial \varphi_1}{\partial \nu(x_0)}(x_0 + \tau(x - x_0))}{-\epsilon \frac{\partial w}{\partial \nu(x_0)}(\lambda, x_0 + \tau(x - x_0))}$$

for some $\tau \in (0, 1)$. Taking a smaller ϵ_0, if necessary, we may suppose that $\frac{\partial w}{\partial \nu(\Pi(x))}(x) < 0$ on $\overline{\omega}$. Then, as above, the quotient in (29) is smaller than some $c_1 > 0$ for λ near $\frac{\lambda_1}{a}$. □

For the second question the answer is delicate. For example we have

PROPOSITION 3. *Suppose* f *to obey the monotone case, that is* $f(t) \geq at$ *for all* t, *and let*

$$l = \lim_{t \to \infty} [f(t) - at] \geq 0.$$

Then

$$\lim_{\lambda \to \frac{\lambda_1}{a}} (\lambda_1 - a\lambda) \|u(\lambda)\|_{L^2(\Omega)} = \frac{\lambda_1}{a} l \int \varphi_1.$$

Proof: Let L_0 be a limit point of $(\lambda_1 - a\lambda)\|u(\lambda)\|_{L^2(\Omega)}$ when $\lambda \to \frac{\lambda_1}{a}$. If we rewrite

$$\text{(10)} \qquad \int \varphi_1[(\lambda_1 - a\lambda)u(\lambda) - \lambda(f(u(\lambda)) - au(\lambda))] = 0$$

in the form

$$\int \varphi_1(\lambda_1 - a\lambda)\|u(\lambda)\|_{L^2(\Omega)} w(\lambda) = \int \lambda\varphi_1(f(u(\lambda)) - au(\lambda)) \tag{30}$$

and we note that the righthand side integrand converges dominated to $\frac{\lambda_1}{a} l\varphi_1$ when $\lambda \to \frac{\lambda_1}{a}$, and that the lefthand side integrand tends to $L_0\varphi_1^2$ $u.$ $\overline{\Omega}$ if $L_0 < \infty$ and to ∞ uniformly in Ω if $L_0 = \infty$ (on an appropriate sequence of λ), we get that

$$L_0 = \frac{\lambda_1}{a} l \int \varphi_1$$

$\square$

It is obvious that the answer is good only when $l > 0$. If $l = 0$ then it shows only that $\|u(\lambda)\|_{L^2(\Omega)}$ grows slower than $\frac{1}{\lambda_1 - a\lambda}$. As we shall see below, in this case the answer depends heavily on f.

EXAMPLE 1. *Let $f(t) = t + \frac{1}{t+2}$ when $t \geq 0$ (defined no matter how for negative t). Then*

$$\lim_{\lambda \to \lambda_1} \sqrt{\lambda_1 - \lambda}\|u(\lambda)\|_{L^2(\Omega)} = \sqrt{\lambda_1 |\Omega|}$$

Proof: With the usual decomposition $u(\lambda) = k(\lambda)w(\lambda)$, if we divide (10) by $\sqrt{\lambda_1 - \lambda}$ we get

$$\int \varphi_1 \sqrt{\lambda_1 - \lambda} k(\lambda) w(\lambda) = \int \frac{\lambda\varphi_1}{\sqrt{\lambda_1 - \lambda} k(\lambda) w(\lambda) + 2\sqrt{\lambda_1 - \lambda}} \tag{31}$$

We claim first that $\liminf\limits_{\lambda \to \lambda_1} \sqrt{\lambda_1 - \lambda} k(\lambda) > 0$. Otherwise, let $\mu_n \to \lambda_1$ be such that $\sqrt{\lambda_1 - \mu_n} k(\mu_n) \to 0$. Then

$$\sqrt{\lambda_1 - \mu_n} k(\mu_n) w(\mu_n) \varphi_1 \to 0 \quad u.\overline{\Omega}$$

and

$$\sqrt{\lambda_1 - \mu_n} k(\mu_n) w(\mu_n) + 2\sqrt{\lambda_1 - \mu_n} \to 0 \quad u.\overline{\Omega},$$

which contradicts (31) for large n.

We shall also prove that $\limsup\limits_{\lambda \to \lambda_1} \sqrt{\lambda_1 - \lambda} k(\lambda) < \infty$. Suppose the contrary. Let $\mu_n \to \lambda_1$ be such that $\sqrt{\lambda_1 - \mu_n} k(\mu_n) \to \infty$. Then the lefthand side of (31) tends to ∞ with n. We shall show that the righthand side remains bounded and the contradiction will conclude the proof. Now $\frac{\varphi_1}{w(\mu_n)}$ is uniformly bounded by some $M > 0$, so that the righthand side integrand is less than $\frac{\lambda_1 M}{\sqrt{\lambda_1 - \mu_n} k(\mu_n)}$, which is bounded.

Let $c \in (0, +\infty)$ be a limit point of $\sqrt{\lambda_1 - \lambda}k(\lambda)$ when $\lambda \to \lambda_1$. Let $\mu_n \to \lambda_1$ be such that $\sqrt{\lambda_1 - \mu_n}k(\mu_n) \to c$ and $\sqrt{\lambda_1 - \mu_n}k(\mu_n) \geq \frac{c}{2}$. Then the lefthand side of (31) tends to c, while the righthand side integrand is dominated by $\frac{2\lambda_1 M}{c}$ and converges *a.e.* to $\frac{\lambda_1}{c}$. Hence $c = \frac{\lambda_1}{c}|\Omega|$ which finishes the proof. $\square$

Note that a similar computation can be made if $f(t) = \sqrt{t^2+1}$.

If $f(t) - at$ decays to ∞ faster than $\frac{1}{t}$ then the behaviour becomes more complicated, as shows

EXAMPLE 2. *Let $f(t) = t + \frac{1}{(t+1)^2}$. Then $\|u(\lambda)\|_{L^2(\Omega)}$ tends to ∞ like no power of $(\lambda_1 - \lambda)$. More precisely,*

i) $\displaystyle\lim_{\lambda \to \lambda_1} (\lambda_1 - \lambda)^\alpha \|u(\lambda)\|_{L^2(\Omega)} = \infty$ *if* $\alpha \leq \dfrac{1}{3}$.

ii) $\displaystyle\lim_{\lambda \to \lambda_1} (\lambda_1 - \lambda)^\alpha \|u(\lambda)\|_{L^2(\Omega)} = 0$ *if* $\alpha > \dfrac{1}{3}$.

Proof: We shall need first some estimations for $\int \frac{1}{\varphi_1}$ and $\int \mathbf{1}_{\{\varphi_1 > \epsilon\}} \frac{1}{\varphi_1}$.

LEMMA 13. *i) There exist positive constants K_1, K_2 and ϵ_1 such that*

$$K_1 |\ln \epsilon| \leq \int \mathbf{1}_{\{\varphi_1 > \epsilon\}} \frac{1}{\varphi_1} \leq K_2 |\ln \epsilon| \quad \textit{for } \epsilon \in (0, \epsilon_1).$$

ii) $\int \frac{1}{\varphi_1} = \infty$.

Proof: ii) follows obviously from *i)*.

i) Let ϵ_0 and ω_0 as in Lemma 12. Let

$$\Phi : \omega_0 \to \partial\Omega \times (-\epsilon_0, \epsilon_0) \ \text{ and } \ \Psi : \partial\Omega \times (-\epsilon_0, \epsilon_0) \to \omega_0$$

be defined by

$$\Phi(x) = (\Pi(x), \langle x - \Pi(x), \nu(x) \rangle) \text{ and } \Psi(x_0, \epsilon) = x_0 + \epsilon \nu(x_0).$$

Then Φ, Ψ are smooth and $\Psi = \Phi^{-1}$, so that if we replace if necessary ϵ_0 with a smaller number, we may suppose that there exist $C_1, C_2 > 0$ such that $0 < C_1 \leq |J(\Psi)| \leq C_2$ on ω_0.

We claim that there exist $C_3, C_4 > 0$ such that

$$C_3 d(x, \partial\Omega) \leq \varphi_1(x) \leq C_4 d(x, \partial\Omega)$$

when $x \in \omega$, if we replace, eventually, ϵ_0 with a smaller number. Indeed, as $\displaystyle\max_{\partial\Omega} \frac{\partial \varphi_1}{\partial \nu} < 0$, we obtain that

$$-C_3 = \sup_{x \in \omega} \frac{\partial \varphi_1(x)}{\partial \nu(\Pi(x))} < 0$$

if ϵ_0 is small enough.

Let $C_4 = \max_{\overline{\Omega}} |\varphi'|$. Then if $x \in \omega$ we get

$$\varphi_1(x) = \varphi_1(x) - \varphi_1(\Pi(x)) = -d(x, \Pi(x)) \frac{\partial \varphi_1(y)}{\partial \nu(\Pi(x))}$$

for some $y \in [x, \Pi(x)]$ and also the desired result.

Take $\epsilon_1 < \min(\inf_{\Omega \setminus \omega} \varphi_1, C_3\ \epsilon_0)$. Now if $\epsilon < \epsilon_1$ then:

$$\int \mathbf{1}_{\{\varphi_1 > \epsilon\}} \frac{1}{\varphi_1} = \int \mathbf{1}_{\{\varphi_1 \geq \epsilon_1\}} \frac{1}{\varphi_1} + \int \mathbf{1}_{\{\epsilon < \varphi_1 < \epsilon_1\}} \frac{1}{\varphi_1}$$

Note that

$$\{\frac{\epsilon}{C_3} < d(x, \partial\Omega) < \frac{\epsilon_1}{C_4}\} \subset \{\epsilon < \varphi_1 < \epsilon_1\} \subset \{\frac{\epsilon}{C_4} < d(x, \partial\Omega) < \frac{\epsilon_1}{C_3}\}$$

and

$$\frac{1}{C_4 d(x, \partial\Omega)} \leq \frac{1}{\varphi_1(x)} \leq \frac{1}{C_3 d(x, \partial\Omega)}$$

there. Then

$$\int \mathbf{1}_{\{\varphi_1 \geq \epsilon_1\}} \frac{1}{\varphi_1} + \frac{1}{C_4} \int \mathbf{1}_{\{\frac{\epsilon}{C_3} < d(x,\partial\Omega) < \frac{\epsilon_1}{C_4}\}} \frac{1}{d(x, \partial\Omega)} \leq$$
$$\leq \int \mathbf{1}_{\{\varphi_1 > \epsilon\}} \frac{1}{\varphi_1} \leq \int \mathbf{1}_{\{\varphi_1 \geq \epsilon_1\}} \frac{1}{\varphi_1} + \frac{1}{C_3} \int \mathbf{1}_{\{\frac{\epsilon}{C_4} < d(x,\partial\Omega) < \frac{\epsilon_1}{C_3}\}} \frac{1}{d(x, \partial\Omega)}$$

It remains to find, for example, $C_5, C_6 > 0$ such that

$$C_5 |\ln\epsilon| \leq I = \int \mathbf{1}_{\{\frac{\epsilon}{C_4} < d(x,\partial\Omega) < \frac{\epsilon_1}{C_3}\}} \frac{1}{d(x, \partial\Omega)} \leq C_6(|\ln\epsilon| + 1)$$

Now with the changement of coordinates $x = \Psi(x_0, \delta)$ we get

$$I = \int_{\partial\Omega \times (\frac{\epsilon}{C_4}, \frac{\epsilon_1}{C_3})} \frac{1}{\delta} |J(\Psi)| ds(x_0) d\delta,$$

so that

$$C_1 |\partial\Omega|\ l\ln \frac{C_4 \epsilon_1}{C_3 \epsilon} \leq I \leq C_2 |\partial\Omega|\ \ln \frac{C_4 \epsilon_1}{C_3 \epsilon}$$

and the desired estimation follows easily. The proof of the Lemma is completed. □

Now in order to prove i) of the Example 2 it is enough to show that

$$\lim_{\lambda \to \lambda_1} (\lambda_1 - \lambda)^{\frac{1}{3}} \|u(\lambda)\|_{L^2(\Omega)} = \infty$$

Suppose that there exist $\mu_n \to \lambda_1$ and $c < \infty$ such that

$$(\lambda_1 - \mu_n)^{\frac{1}{3}} k_n \to c, \quad \text{where } k_n = \|u(\mu_n)\|_{L^2(\Omega)}$$

If we divide (10) written with $\lambda = \mu_n$ by $(\lambda_1 - \mu_n)^{\frac{2}{3}}$ we get

$$\int \varphi_1 (\lambda_1 - \mu_n)^{\frac{1}{3}} k_n w_n = \lambda \int \frac{\varphi_1}{(\lambda_1 - \mu_n)^{\frac{2}{3}} (k_n w_n + 1)^2} \tag{32}$$

where $w_n = \frac{1}{k_n} u(\mu_n)$.

If $c = 0$ then the lefthand side in (32) tends to 0, while the second one to ∞. Hence $c \in (0, \infty)$. The fact that $k_n \to \infty$ implies that for each $\epsilon > 0$, $2k_n w_n + 1 < \epsilon k_n^2$, for large n, so that the righthand side of (32) is larger that

$$\frac{\lambda}{2c^2} \int \frac{\varphi_1}{\varphi_1^2 + \epsilon}$$

for n big enough to have $(\lambda_1 - \mu_1)^{\frac{2}{3}} k_n^2 < 2c^2$. Since the limit of the lefthand side is c, we get that

$$c \geq \frac{\lambda_1}{2c^2} \int \frac{\varphi_1}{\varphi_1^2 + \epsilon}$$

for all $\epsilon > 0$. Leting $\epsilon \to 0$ we obtain $c = \infty$, the desired contradiction.

ii) Suppose the contrary. Then there exist $\alpha > \frac{1}{3}$, $\mu_n \to \lambda_1$, $c \in (0, +\infty]$ such that $(\lambda_1 - \lambda)^\alpha k_n \to c$, where $k_n = \|u(\mu_n)\|_{L^2(\Omega)}$.

Let $\beta = 3\alpha - 1 > 0$. Then (10) with $\lambda = \mu_n$ divided by $(\lambda_1 - \lambda)^{1-\alpha}$ gives

$$\int \varphi_1 (\lambda_1 - \mu_n)^\alpha k_n w_n = \lambda \int \frac{\varphi_1}{(\lambda_1 - \mu_n)^{2\alpha - \beta} (k_n w_n + 1)^2} \; (= I_n) \tag{33}$$

The limit of the lefthand side is $c \in (0, +\infty]$. I_n can be estimated as follows:

$$I_n = \int ... = \int \mathbf{1}_{\{\varphi_1 < \lambda_1 - \mu_n\}} ... + \int \mathbf{1}_{\{\varphi_1 \geq \lambda_1 - \mu_n\}} ... = J_n + K_n$$

Now

$$0 < J_n \leq \int \frac{\lambda_1 - \mu_n}{(\lambda_1 - \mu_n)^{2\alpha - \beta}} = (\lambda_1 - \mu_n)^\alpha |\Omega| \to 0$$

while

$$0 < K_n \leq \frac{M(\lambda_1 - \mu_n)^\beta}{c^2} \int \mathbf{1}_{\{\varphi_1 \geq \lambda_1 - \mu_n\}} \frac{1}{\varphi_1},$$

where $M = \sup_n \max \frac{w_n^2}{\varphi_1^2} < \infty$ (as shows the proof of the Proposition 2).

Lemma 13 shows that the last expression is $O((\lambda_1-\mu_n)^\beta|\ln(\lambda_1-\mu_n)|)$, that is it tends to zero with n.

In the non-monotone case $\|v(\lambda)\|_{L^2(\Omega)}$ grows faster to ∞. We have

PROPOSITION 4. *Let f obey the non-monotone case and let*

$$\lim_{t\to\infty}[f(t)-at]=l\in[-\infty,0).$$

Then

$$\lim_{\lambda\to\frac{\lambda_1}{a}}(\lambda_1-a\lambda)\|v(\lambda)\|_{L^2(\Omega)}=l$$

The proof is identical to that of the preceding Proposition.

The result is good only when $l\in\mathbf{R}$. When $l=-\infty$, we give an example.

EXAMPLE 3. *If $f(t)=t+2-\sqrt{t+1}$, then*

$$\lim_{\lambda\to\lambda_1}(\lambda-\lambda_1)^2\|v(\lambda)\|_{L^2(\Omega)}=(\int\varphi_1\sqrt{\varphi_1})^2$$

Proof: If we multiply (10) by $\lambda-\lambda_1$ we get

$$\int\varphi_1(\lambda-\lambda_1)\sqrt{k(\lambda)}\sqrt{w(\lambda)}[\lambda-(\lambda-\lambda_1)\sqrt{k(\lambda)}\sqrt{w(\lambda)}]= \tag{34}$$

$$=2\lambda(\lambda-\lambda_1)\int\varphi_1-\lambda\int\varphi_1[\sqrt{(\lambda-\lambda_1)^2k(\lambda)w(\lambda)+(\lambda-\lambda_1)^2}-\sqrt{(\lambda-\lambda_1)^2k(\lambda)w(\lambda)}]$$

where $k(\lambda),w(\lambda)$ are as usual. We prove first that $\limsup\limits_{\lambda\to\lambda_1}(\lambda-\lambda_1)^2k(\lambda)<\infty$. Suppose there exist $\mu_n\to\lambda_1$ such that $(\mu_n-\lambda_1)^2k(\mu_n)\to\infty$. Then the righthand side of (34) tends to 0, while the lefthand side is, for a suitable choice of $C_1,C_2>0$, less than

$$C_1(\lambda-\lambda_1)\sqrt{k(\lambda)}-C_2(\lambda-\lambda_1)^2k(\lambda)$$

so it tends to $-\infty$.

Suppose now that

$$\liminf_{\lambda\to\lambda_1}(\lambda-\lambda_1)^2k(\lambda)=0. \tag{35}$$

The last integral in (34) is positive, so that (34) gives

$$\int\varphi_1\sqrt{k(\lambda)}\sqrt{w(\lambda)}[\lambda-(\lambda-\lambda_1)\sqrt{k(\lambda)}\sqrt{w(\lambda)}]\le2\lambda\int\varphi_1 \tag{36}$$

But the assumption (35) makes the lefthand side of (36) to tend to ∞ for a suitable λ. The contradiction shows that (35) is false.

Now let $c \in (0, +\infty)$ be any limit point of $(\lambda - \lambda_1)^2 k(\lambda)$ when $\lambda \to \lambda_1$. Then (34) shows that $c = (\int \varphi_1 \sqrt{\varphi_1})^2$.

All other functions we have tested behaved well in the sense that $\|v(\lambda)\|_{L^2(\Omega)} \sim Cg(\frac{1}{\lambda - \lambda_1})$ where g is the inverse of the antiderivative of

$$[0, +\infty) \ni t \longmapsto \frac{1}{at + f(0) + 1 - f(t)}$$

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PERIODIC SOLUTIONS OF THE EQUATION

$-\Delta v = v(1-|v|^2)$ IN R AND $\mathbf{R^2}$

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1. Introduction

We study in this paper the existence of periodic functions $v : \mathbf{R} \to \mathbf{C}$ which satisfy the equation

$$-v'' = v(1-|v|^2)\,. \tag{1}$$

As observed in [BMR], the functions

$$Ae^{ikx} \quad , \text{ where } \; k \in \mathbf{R},\; A \in \mathbf{C},\; |A|^2 + k^2 = 1\,, \tag{2}$$

are such solutions.

For fixed T, we also study the number of solutions of (1) with principal period T. The problem is that (1) has too many solutions, that is, if v is a solution, then

$$x \longmapsto \alpha v(x_0 \pm x) \tag{3}$$

is also a solution if $|\alpha| = 1$ and $x_0 \in \mathbf{R}$. In order to avoid such a redundance, we shall first obtain a "canonical form" of solutions of (1). Namely, let V be a periodic solution of (1). We may suppose that $x = 0$ is a maximum point for $|V|^2$. Then one can find $\epsilon \in \{-1, +1\}$ and $\alpha \in \mathbf{C}$, $|\alpha| = 1$ such that

$$x \overset{v}{\longmapsto} \alpha V(\epsilon x)$$

satisfies, apart (1), the conditions

$$\begin{cases} v_1(0) = a > 0 \\ v_1'(0) = 0 \\ v_2(0) = 0 \\ v_2'(0) = b \geq 0 \quad , \end{cases} \tag{4}$$

where $v = v_1 + iv_2$ and $a = \max|v|$. It is obvious that the system (1)+(4) gives all the geometrically distinct solutions of (1), that is solutions that cannot be obtained one from another by the procedure (3).

In what follows, we shall simply write "T-periodic solutions" instead of "solutions of principal period T". Our first result concerns the existence and the multiplicity of "T-periodic solutions".

2. The main result

Our main result is the following

Theorem. *i) If $T \leq 2\pi$, there are no T-periodic solutions.*

ii) If $T > 2\pi$, there is exactly one real solution v of (1)+(4), that is a solution for which $v_2 \equiv 0$. Moreover, v depends analytically on T.

iii) There is some $T_1 > 2\pi$ such that, for $2\pi < T \leq T_1$, (1)+(4) has no other T-periodic solutions apart those given by ii) above and (2), for $k = \frac{2\pi}{T}$, $A = \sqrt{1-k^2}$.

iv) For $T > T_1$, (1)+(4) has other T-periodic solutions apart these two.

v) For each T, the number of T-periodic solutions is finite.

vi) For large T, (1)+(4) has at least

$$\frac{5}{8}T^2 + O(T \log T)$$

T-periodic solutions.

Remark: In fact, we shall find <u>all</u> the solutions of (1)+(4). More precisely, we shall exhibit a set $\Omega = \overline{\Omega} \subset \mathbf{R}^2$ such that, roughly speaking,

i) if $(a,b) \notin \Omega$, then the solution of (1)+(4) has a finite life time for positive <u>or</u> negative x.

ii) if $(a,b) \in \partial\Omega$, we obtain the solutions given by (2) or ii) of the Theorem.

iii) if $(a,b) \in \mathrm{Int}\Omega$, then $v \neq 0$, v has a global existence, $|v|$ and $\frac{\mathrm{d}}{\mathrm{d}x}\frac{v}{|v|}$ are periodic functions. For such (a,b), if T_0 is the principal period of $|v|$ and φ is (globally) defined such that $v = e^{i\varphi}|v|$, then v is periodic if and only if $\varphi(T_0) - \varphi(0) \in \pi$. Given $q = \frac{m}{n} \in$, $q > 0$, $(m,n) = 1$, the set

$$\{(a,b) \in \mathrm{Int}\ \Omega\ ;\quad \varphi(T_0) - \varphi(0) = \pi q\}$$

is a smooth curve, which for example can be parametrized as $(a, b(a))$, $a \in (a_0, 1)$, where a_0 is depending on q. If $T_0(a)$ denotes the principal period of $|v|$ for the initial datae $(a, b(a))$, then $\lim\limits_{a\nearrow 1} T_0(a) = \infty$ and this curve raises a smooth curve of periodic solutions of (1)+(4), with principal periods $T(a) = nT_0(a)$ (if m is even) or $T(a) = 2nT_0(a)$ (if m is odd).

Actually, the diagram of bifurcation of the distinguished solutions is given by Picture 1.

For the instant, we do not know whether the curves q =const. are like a) or like b) in Picture 1. In other words, we do not know whether T increases or not along these curves. If the first possibility holds, the minimum number of solutions given by (38) is the exact one. After the proof of the theorem, we shall give a sufficient condition for this happens (see the Remarks following the proof).

Finally, the last paragraph is devoted to the existence, in the whole $\mathbf{R}^2$, of 2-periodic solutions which are geometrically distinct to the real ones. Some existence and non-existence results are obtained.

3. Proof of Theorem

Let us note first that

$$a \leq 1\,. \tag{5}$$

Suppose the contrary. Let $M > 1$ be such that

$$\min|v| < M < \max|v|\,.$$

Let I be an interval such that $|v| > M$ in I and $|v| = M$ on ∂I. (Note that such an interval is necessarily finite). Since

$$(|v|^2)'' \geq 2|v|^2(|v|^2 - 1) > 0$$

in I, it follows that $|v| \leq M$ in I, which contradicts our choice of I.

Next we shall prove that

$$b^2 \leq a^2(1 - a^2)\,. \tag{6}$$

Indeed, for small x we have

$$v_1(x) = a - \frac{a(1-a^2)}{2}x^2 + O(x^3)\,,$$

$$v_2(x) = bx + O(x^3)\,,$$

so that (6) follows from the fact that $x = 0$ is a local maximum.

Now let

$$\Omega = \{(a,b) \in (0,1] \times [0,1]; \quad b^2 \leq a^2(1-a^2)\}\,.$$

We have obtained that if (1)+(4) raises a non-null periodic solution such that $x = 0$ is a local maximum, then necessarily $(a,b) \in \Omega$.

We shall first study the case $(a,b) \in \partial\Omega$.

Case 1 If $b = a\sqrt{1-a^2}$, it follows that

$$v(x) = ae^{ikx} \qquad \text{, where } \; k = \sqrt{1-a^2}\,.$$

Indeed, (2) provides a solution for (1)+(4) in this case.

Case 2 If $b = 0$, one gets easily that $v_2 = 0$. If $a = 1$, we get the trivial solution $v(x) \equiv 1$, so that in what follows we shall assume that $a \in (0,1)$.

Note first that v_1 cannot be positive (negative) into an infinite interval if v is periodic. For, otherwise, v_1 would be a periodic concave (convex) function, that is a constant function.This is impossible for our choice of a and b.

Let x_1, x_2 be two consecutive zeros of v_1. We may suppose that $v(x) > 0$ if $x_1 < x < x_2$, so that $v'(x_1) > 0$, $v'(x_2) < 0$. If x_3 is the smallest $x > x_2$ such that $v(x_3) = 0$, it follows that $v(x) < 0$ if $x_2 < x < x_3$.

If we prove the fact that $x_2 - x_1 > \pi$, it will also follow that $x_3 - x_1 > 2\pi$ and that there is no $x \in (x_1, x_3)$ such that $v(x) = 0$ and $v'(x) > 0$. We will get that the principal period of v must be $> 2\pi$. This will be done in

Lemma 1. *Let $f : \mathbf{R} \to [0,1]$ be such that the set $\{x;\ f(x) = 0 \quad or \quad f(x) = 1\}$ contains only isolated points. Let v be a real function such that $v(x_1) = v(x_2) = 0$, and $v(x) > 0$ in $(x_1,\, x_2)$. If, for $x \in [x_1, x_2]$,*

$$-v'' = vf\,, \tag{7}$$

then $x_2 - x_1 > \pi$.

Proof. We may assume that $x_1 = 0$. Multiplying (7) by $\varphi(x) := \sin\dfrac{\pi x}{x_2}$ and integrating by parts, we obtain that

$$\int_0^{x_2} v\varphi > \int_0^{x_2} vf\varphi = (\frac{\pi}{x_2})^2 \int_0^{x_2} v\varphi\,,$$

that is $x_2 > \pi$. □

Incidentally, this proves i) of the Theorem.

Returning to the Case 2, we shall explicitely integrate (1)+(4) as one usually does for the Weierstrass Elliptic Functions. Multiplying (1) by v_1', we find

$$v_1'^2 = -v_1^2 + \frac{1}{2}v_1^4 + a^2 - \frac{1}{2}a^4 \,. \tag{8}$$

It follows that, as far as the solution of (1)+(4) exists, we have $|v_1| \leq a$ and $|v_1'| \leq$ $\leq \sqrt{a^2 - \frac{1}{2}a^4}$. Hence the solution of (1)+(4) is globally defined.

Note that $v_1'(0) = 0$, $v_1''(0) < 0$, so that v_1 decreases for small $x > 0$. Moreover, $v_1'(x) < 0$ for $0 < x < \tau$, where

$$\tau = \sup\{x > 0; \quad v_1(y) > 0 \quad \text{for all} \;\; 0 < y < x\} \,.$$

Indeed, suppose the contrary. Then, taking (8) into account, we obtain the existence of some $\tau_0 > 0$ such that $v_1(\tau_0) = a$, $\tau_0 < \tau$. If we consider the smallest $\tau_0 > 0$ such that the above equality occurs, we have $v_1(x) < a$ if $0 < x < \tau_0$. Since $v_1(0) = v_1(\tau_0) = a$, it follows that there exists some $0 < \tau_1 < \tau_0$ such that $v_1'(\tau_1) = 0$, which is the desired contradiction. Hence we have

$$v_1' = -\sqrt{a^2 - \frac{1}{2}a^4 - v_1^2 + \frac{1}{2}v_1^4} < 0 \qquad \text{in} \;\; (0, \tau) \,. \tag{9}$$

It follows that, if $0 < x < \tau$, then

$$\int_{v(x)}^{a} \frac{1}{\sqrt{\frac{1}{2}t^4 - t^2 + a^2 - \frac{1}{2}a^4}} dt = x \,,$$

which gives

$$\tau = \int_0^a \frac{dt}{\sqrt{\frac{1}{2}t^4 - t^2 + a^2 - \frac{1}{2}a^4}} := \tau(a) \,. \tag{10}$$

From (1), we obtain $v_1(\tau + x) = -v_1(\tau - x)$, $v_1(2\tau - x) = -v_1(x)$, $v_1(4\tau + x) = v_1(x)$, so it is easy to see that v is periodic of principal period $T(a) = 4\tau(a)$.

Now (10) can be rewritten as

$$\tau(a) = \int_0^1 \frac{1}{\sqrt{(1-\xi^2)[1 - \frac{a^2}{2}(1+\xi^2)]}} d\xi \,, \tag{11}$$

so that τ increases with a and

$$\lim_{a \searrow 0} \tau(a) = \frac{\pi}{2}, \quad \lim_{a \nearrow 1} \tau(a) = +\infty \,.$$

Since $\tau'(a) > 0$, it follows that the mapping

$$T(a) \longmapsto a := a(T)$$

is analytic, so that ii) is completely proved. Moreover,

$$\lim_{T \searrow 2\pi} a(T) = 0 \quad \text{and} \quad \lim_{T \nearrow \infty} a(T) = 1,$$

so that the diagram of "real" solutions is that depicted in Picture 1.

Next we return to the points (a, b) which are interior to Ω.

<u>Case 3</u> Let $(a, b) \in \mathrm{Int}\Omega$.

Write, for small x,

$$v(x) = e^{i\varphi(x)} w(x) \tag{12}$$

with $\varphi(0) = 0$ and $w > 0$.

One can easily see that w satisfies

$$-w'' = w(1 - w^2) - \frac{a^2 b^2}{w^3} \tag{13}$$

and

$$\begin{cases} w(0) = a \\ w'(0) = 0\,, \end{cases} \tag{14}$$

while φ is given by

$$\varphi' = \frac{ab}{w^2}, \qquad \varphi(0) = 0\,. \tag{15}$$

Hence, if the system (13)+(14) has a global <u>positive</u> solution, it follows that (12) is global. Moreover, if w is periodic of period T_0, then

$$v(nT_0 + x) = e^{in\varphi(T_0)} e^{i\varphi(x)} w(x) \quad \text{for} \;\; 0 \le x < T_0, \;\; n = 0, 1, ... \tag{16}$$

so that (1)+(4) gives a periodic solution if and only if $\varphi(T_0) \in \pi$.

We shall prove the global existence in

Lemma 2. *If* $(a, b) \in \mathrm{Int}\;\Omega$, *then (13)+(14) have a global positive periodic solution.*

Proof. Note that the assumption made on (a,b) implies that $w''(0) < 0$, so that, multiplying as above (13) by w', we obtain , for small $x > 0$,

$$w'^2 = -w^2 + \frac{1}{2}w^4 - \frac{a^2b^2}{w^2} + a^2 - \frac{1}{2}a^4 + b^2 \tag{17}$$

and

$$w' = -\sqrt{-w^2 + \frac{1}{2}w^4 - \frac{a^2b^2}{w^2} + a^2 - \frac{1}{2}a^4 + b^2}\,. \tag{18}$$

Now (17) implies that w and w' are bounded as far as the solution exists and, moreover, that

$$\inf\{w(x);\quad w \text{ exists}\} > 0\,.$$

It follows that w is a global solution. Let

$$\tau = \sup\{x > 0;\quad w'(y) < 0 \quad \text{for all} \;\; 0 < y < x\}\,.$$

Note that (18) is valid if $0 < x < \tau$.

Let c be the only root of

$$f(x) := -x^2 + \frac{1}{2}x^4 - \frac{a^2b^2}{x^2} + a^2 - \frac{1}{2}a^4 + b^2 = 0$$

which is positive and inferior to a.

Since $f(x) < 0$ if $0 < x < c$ or $x > a$, x close to a, it follows from (17) that

$$c \le w(x) \le a \qquad \text{for all} \;\; x \in \mathbf{R}\,. \tag{19}$$

Claim 1. $\lim\limits_{x \nearrow \tau} w(x) = c$.

Proof of Claim 1. If $\tau < \infty$, it follows that $w'(\tau) = 0$. Now (17) together with the definitions of τ and c show that $w(\tau) = c$. If $\tau = \infty$, then we have $\lim\limits_{x \to \infty} w(x) \ge c$. If we would have $\lim\limits_{x \to \infty} w(x) > c$, there would exist a constant $M > 0$ such that $w'(x) \le -M$ for each $x > 0$. The latest inequality contradicts (19) for large x.

As we did before, for $0 < x < \tau$, (18) gives

$$x = \int_{w(x)}^{a} \frac{1}{\sqrt{-t^2 + \frac{1}{2}t^4 - \frac{a^2b^2}{t^2} + a^2 - \frac{1}{2}a^4 + b^2}} dt\,, \tag{20}$$

so that

$$\tau = \int_c^a \frac{1}{\sqrt{-t^2 + \frac{1}{2}t^4 - \frac{a^2b^2}{t^2} + a^2 - \frac{1}{2}a^4 + b^2}} dt < \infty . \tag{21}$$

It follows by a reflection argument that $w(2\tau) = w(0) = a$, $w'(2\tau) = w'(0) = 0$, so that w is (2τ)-periodic. □

Next, in order to make simpler the computations that follow, it is useful to replace the (a, b)-coordinates into other ones, by associating to (a, b) the point (A, C), where $A = a^2$, $C = c^2$ with a, c as above. This changement of coordinates maps Ω analytically into

$$\omega := \{(A, C); \quad 0 < C < A, \ 2A + C < 2\}$$

(see Picture 2).

It follows from the above discussion that to each $(A, C) \in \omega$ it corresponds a solution (w, φ) of (13)-(15) such that w and φ' are periodic of period given by (after a suitable change of variables)

$$T_0 = T_0(A, C) = 2\sqrt{2} \int_0^\infty \frac{1}{\sqrt{(y^2 + 1)[(2 - 2A - C)y^2 + (2 - A - 2C)]}} dy . \tag{22}$$

Moreover, $\varphi(0) = 0$ and

$$\varphi(T_0) = \sqrt{2AC(2 - A - C)} \int_0^{\tau(A,C)} \frac{1}{w^2(y)} dy , \tag{23}$$

where $\tau(A, C) = \frac{1}{2}T_0(A, C)$.

Now the change of variables $w(y) = t$ yields, with $\varphi(A\,, C) := \varphi(T_0\,(A\,, C))$,

$$\varphi(A, C) = \sqrt{2AC(2 - A - C)} \int_0^\infty \sqrt{\frac{y^2 + 1}{(2 - 2A - C)y^2 + (2 - A - 2C)}} \cdot \frac{1}{Ay^2 + C} dy , \tag{24}$$

and (22), (24) show that $(A, C) \longmapsto (T_0, \varphi)$ is an analytic map. Moreover, (22) gives that

$$T_0 > \pi, \qquad \lim_{(A,C)\to(0,0)} T_0(A, C) = \pi, \qquad \inf_{|(A,C)|\geq\varepsilon>0} T_0(A, C) > \pi . \tag{25}$$

A lower estimate for φ will be given in

Lemma 3. $\varphi > \dfrac{\pi}{\sqrt{2}}$ *and* $\displaystyle\lim_{(A,C)\to(0,0)} \varphi(A, C) = \frac{\pi}{\sqrt{2}}$.

Proof. If we put $y = \sqrt{\frac{C}{A}}z$ in (24), we obtain

$$\varphi(A,C) = \sqrt{2(2-A-C)}\int_0^\infty \sqrt{\frac{Cz^2+A}{C(2-2A-C)z^2+A(2-A-2C)}}\frac{1}{z^2+1}dz\,, \tag{26}$$

so that the second assertion follows from the Lebesgue Dominated Convergence Theorem.

For the first one, it is enough to show that for given $0 < k < 1$, the function

$$(0,\frac{2}{k+2}) \ni A \overset{\psi}{\longmapsto} \varphi(A,kA)$$

is increasing.

After a short computation, we find that

$$\psi'(A) = \frac{2k}{\sqrt{2-(k+1)A}}\int_0^\infty \sqrt{\frac{ky^2+1}{[k(2-(k+2)A)y^2+(2-(2k+1)A)]^3}}dy > 0\,. \tag{27}$$

□

Incidentally, this shows that φ has no critical points and that the level curves φ=const. are analytic and can be parametrized as

$$(A(k),kA(k))\,. \tag{28}$$

Lemma 4. $\lim_{A\nearrow\frac{2}{k+2}} \psi(A) = \infty.$

Proof. It follows from (26) that

$$\psi(A) > \sqrt{2(2-\frac{2(k+1)}{k+2})}\int_0^\infty \sqrt{\frac{kx^2+1}{k(2-(k+2)A)z^2+2-(2k+1)A}}\frac{dz}{z^2+1}\,,$$

and the last integral tends monotonically to $+\infty$ by the Beppo Levi Theorem. □

From the above Lemma, we obtain that the parametrization (28) is valid for $k \in (0,1)$. Moreover, (27) shows that the mapping

$$k \longmapsto A(k) \tag{29}$$

is analytic. Of course,the level line φ =const. is non-void if and only if const.$> \frac{\pi}{\sqrt{2}}$. This will be assumed in the sequel. We shall prove that (29) provides a decreasing mapping.

Indeed, if we consider now ψ as $\psi(A,k)$, then it follows from (27) that $\dfrac{\partial\psi}{\partial A}$ increases with k. Hence, if $k_1 < k_2$, then

$$\psi(A,k_1) < \psi(A,k_2)\,,$$

that is $A(k)$ decreases with k.

We obtain the existence of

$$\lim_{k\nearrow 1} A(k) := A_0 \quad \text{and} \quad \lim_{k\searrow 0} A(k) := A_1 > A_0\,,$$

From Lemma 3, $A_0 > 0$.

Claim 2. $A_1 = 1$.

Proof of Claim 2. It follows from (26) and the Lebesgue Dominated Convergence Theorem that

$$\lim_{(A,C)\to(A_2,0)} \varphi(A,C) = \frac{\pi}{\sqrt{2}} \quad \text{if } 0 < A_2 < 1\,,$$

so that, taking Lemma 3 into account, we obtain that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varphi(A,C) < \frac{\pi}{\sqrt{2}} + \varepsilon$$

if $0 < A < 1-\delta$, $0 < C < \delta$. This completes the proof of the claim. □

At this stage of the proof, we know that the level lines φ=const. are analytic, all of them "end" at (1,0) and "begin" at (A_0, A_0) for some suitable $0 < A_0 < 1$, A_0 depending on the constant. Moreover, if $q_1 < q_2$, the line $\varphi = q_1$ lies below the line $\varphi = q_2$ (see Picture 3).

Now A_0 can be found implicitely, because φ can be extended by continuity on the line segment MN. This shows that

$$q = \varphi(A_0, A_0) = \frac{\pi}{2}\sqrt{\frac{1-A_0}{2-3A_0}}\,,$$

that is

$$A_0 = A_0(q) = \frac{8q^2 - \pi^2}{12q^2 - 3\pi^2}\,. \tag{30}$$

Returning to the proof of the theorem, note that iii) and iv) follow easily from the above calculation. Indeed, for small A and C, if $\varphi(A,C) = \pi\dfrac{m}{n}$ is a rational multiple of π, then $n \geq 4$, so that, taking into account the fact that $T_0(A,C) \geq \pi$, it follows that for small A the period of v is at least 4π. Now the existence of T_1 follows from (25).

In order to prove v), note that the level line $\varphi = q$ contains a T-periodic solution if and only if

$$(31)\qquad \begin{cases} q = \pi \dfrac{m}{n},\ (m,n)=1 \quad \text{and there exists } (A,C) \quad \text{on the level line} \\ \text{such that} \qquad T_0(A,C) = \begin{cases} \frac{T}{n}, & \text{if } m \text{ is even} \\ \frac{T}{2n}, & \text{if } m \text{ is odd} \end{cases} \end{cases}.$$

We shall prove that

$$(32)\qquad \lim_{2A+C \nearrow 2} T_0(A,C) = \infty\,.$$

Suppose (32) proved for the moment. Obviously, if $\varphi(A_n,\, C_n) \;\to\; \infty$, then $2A_n + C_n \to 2$. It follows from (32) that, for q large enough, $T_0(A,\, C) > T$ if $(A,\, C)$ is on the level line $\varphi = q$, so that (31) cannot hold for such q. Hence, in order to prove v) it remains to show that, for given q, T_0 , the set

$$\mathcal{M} = \{(A,C);\;\; \varphi(A,C) = q,\ T_0(A,C) = T_0\}$$

is finite.

Let

$$\mathcal{C}_1 = \{(A,C);\;\; \varphi(A,C) = q\}\,.$$

Since $\mathcal{C}_1$ is an analytic curve, $\mathcal{M}$ is finite provided that (1,0) and $(A_0(q), A_0(q))$ are not cluster points of $\mathcal{M}$. For (1,0), this follows from the fact that, according to (32), $T_0(A,C)$ approaches $+\infty$ as A approaches 1 along $\mathcal{C}_1$. In particular, $T_0(A,C)$ is not constant along $\mathcal{C}_1$. In order to see what happens in $(A_0(q), A_0(q))$, we perform the following trick: let

$$\omega_1 = \omega \cup \{(C,A);\;\; (A,C) \in \omega\} \cup \{(A,A);\;\; 0 < A < 1\}$$

(see Picture 4).

Obviously, (24) extends φ to an analytic function φ_1 in ω_1. The change of variables $z = \dfrac{1}{y}$ in (24) shows that $\varphi(A,C) = \varphi(C,A)$. Note also that (27) continues to hold for $k = 1$. This shows that φ_1 has no critical points and that $T_0(A,C)$ tends to $+\infty$ at the both ends of φ_1=const. Hence, φ can assume the same value only a finite number of times.

All it remains to do is

Proof of (32). Let $A_n < 1$, $0 < C_n < 1$ be such that $2A_n + C_n \nearrow 2$. Then

$$(33)\qquad T_0(A_n, C_n) > 2\sqrt{2} \int_0^\infty \frac{dy}{\sqrt{(y^2+1)[(2-2A_n-C_n)y^2+2]}}\,,$$

and the right hand side of (33) tends to $+\infty$ from the Beppo Levi Theorem. $\square$

The proof of v) is completed.

Next we return to the proof of vi). Take $q = \pi\frac{m}{n}$, $(m,n) = 1$, $\frac{m}{n} > \frac{1}{\sqrt{2}}$. Then the level line $\varphi = q$ is nonempty and smooth. If we put

$$(34) \qquad T_0(q) = 2\sqrt{2}\int_0^{\infty} \frac{dy}{\sqrt{(y^2+1)[(2-3A_0(q))y^2+(2-3A_0(q))]}} = \pi\sqrt{\frac{24q^2-6\pi^2}{16q^2-5\pi^2}}\,,$$

it follows from (32) that, along $\varphi = q$, T_0 assumes all the values between $T_0(q)$ and $+\infty$. We obtain that, for fixed T, (1)+(4) has at least one T-periodic solution corresponding to each q such that

$$(35) \qquad q = \pi\frac{m}{n}, \quad (m,n) = 1, \quad \frac{m}{n} > \frac{1}{\sqrt{2}}, \quad T_0(q) < \begin{cases} \frac{T}{n}, & \text{if } m \text{ is even} \\ \frac{T}{2n}, & \text{if } m \text{ is odd}\ . \end{cases}$$

Hence it suffices to count, for large T, the number of elements of $A \cup B$, where

$$(36) \quad A = \{(m,n);(m,n) = 1, m \text{ is even}, \frac{m}{n} > \frac{1}{\sqrt{2}}, 24m^2n^2 - 6n^4 < (16m^2n^2 - 5n^2)\pi^2T^2\}$$

and

$$(37) \quad B = \{(m,n);(m,n) = 1, m \text{ is odd}, \frac{m}{n} > \frac{1}{\sqrt{2}}, 96m^2n^2 - 24n^4 < (16m^2n^2 - 5n^2)\pi^2T^2\}\,.$$

Note that

$$A \cup B \supset \{(m,n); \quad (m,n) = 1,\ m \geq n,\ m \leq \sqrt{\frac{5}{24}}\pi T\}\,.$$

It follows that there are at least

$$(38) \qquad \sum_{1 \leq m \leq \sqrt{\frac{5}{24}}\pi T} \Phi(m)$$

solutions, where Φ is the Euler's Function. Now a Theorem of Mertens (see [Ch]) asserts that the sum in (38) is

$$(39) \qquad \frac{5}{8}T^2 + O(T\log T)\,.$$

The proof of the Theorem is completed. $\square$

Remarks: 1) It is obvious that (38) does not provide an accurate estimate. On the other hand, one may see that the number of elements of $A \cup B$ is $O(T^2)$.

2) (35) counts <u>all</u> the T-periodic solutions if and only if T_0 increases along φ=q=const. as far as A increases from $A_0(q)$ to 1. A sufficient condition is that $(A,C) \longmapsto (T_0(A,C), \varphi(A,C))$ is a local diffeomorfism. This relies on the following fact: let ω be an open connected set of $\mathbf{R}^2$ and $f : \omega \to \mathbf{R}^2$ a <u>local</u> diffeomorfism. If the level lines f_2=const. are connected, then $f : \omega \to f(\omega)$ is a <u>global</u> diffeomorfism.

3) It follows from the proof that the diagram of bifurcation is, indeed, as in Picture 1. For example, the level line $\varphi = q$, $q = \pi \dfrac{m}{n}$, raises a branch of periodic solutions which starts from a solution of the form (2). Note that, on a level line, the solutions oscillate more and more as $A \nearrow 1$, in the sense that $\max |v|$ and $\min |v|$ approach 1 and 0 as A approaches 1. It is also easy to see that, in Picture 1, the points $T_1, T_2, T_3, ...$ are isolated.

4) One may prove that, if $a = \max |v|$ for a T-periodic solution, then

i) $a^2 + (\dfrac{2\pi}{T})^2 = 1$ if v is given by (b);

ii) $a^2 + (\dfrac{2\pi}{T})^2 > 1$ if v is a real solution;

iii) $a^2 + (\dfrac{2\pi}{T})^2 < 1$ if v is a "complex" solution.

5) We have seen that the solution of (1)+(4) is globally existent if $(A,C) \in \overline{\omega}$. The same happens if $(A,C) \in \overline{\omega_1}$. There is nothing surprising in this, because starting with some $(A,C) \in \omega_1 \setminus \omega$ means considering the "canonical form" of (1) with $x = 0$ a local <u>minimum</u>, this time.

Let Ω_1 be the inverse image of ω_1 with respect to the mapping $(a,b) \longmapsto (A,C)$. Considering some point (a,b), $a \geq 0$, $b \geq 0$ such that $(a,b) \notin \overline{\Omega_1}$, it is easy to carry out once again (13)-(21) in order to prove that this time v has a finite left <u>or</u> right life time.

4. Existence of non-trivial periodic solutions in $\mathbf{R}^2$

We are concerned with the existence of double periodic solutions, that is of functions $u : \mathbf{R}^2 \to \mathbf{C}$ solutions of

$$-\Delta u = u(1-|u|^2), \quad u \in L^2_{loc}(\mathbf{R}^2)\,, \tag{40}$$

such that there exist $\omega_1, \omega_2 \in \mathbf{R}^2$ linearly independent with

$$u(x+\omega_j) = u(x), \qquad j = 1, 2\,. \tag{41}$$

Of course, we have already obtained such solutions: take $\omega_1 = (2T, 0)$ with $T > \pi$, ω_2 arbitrary and u a $2T$-periodic real solution. Even simpler, one may take u=const., $|u| = 0$ or 1.

Therefore, we shall look for non-trivial solutions, that is solutions enjoying the property

$$\left\{ \begin{array}{c} \text{there is no } v : \mathbf{R} \to \mathbf{C} \text{ solution of (1) such that} \\ u(x) = v(\alpha_1 x_1 + \alpha_2 x_2) \text{ for some } \alpha \in \mathbf{C},\ |\alpha| = 1\,. \end{array} \right. \tag{42}$$

We start with a non-existence result.

Proposition 1. *If $|\omega_1|, |\omega_2|$ are small enough, all the solutions of (40)-(41) are constant.*

We shall use in the proof

Lemma 5. *Let u be a solution of (40)-(41). Then $|u| \le 1$ (so that u is smooth).*

Proof of Lemma 5. We follow an idea from [BMR]. It follows easily from (40) that $u \in H^1_{\rm loc}(\mathbf{R}^2)$. Let

$$P = \{\lambda\omega_1 + \mu\omega_2;\quad 0 \le \lambda \le 1, 0 \le \mu \le 1\}\,.$$

Let φ be a $C_0^\infty(\mathbf{R}^2)$-function such that $\varphi \ge 0$, $\varphi = 1$ in a neighborhood of 0, and $\varphi_n(x) = \dfrac{1}{n^2}\varphi(\dfrac{x}{n})$ for $n = 1, 2, ...$

Multiplying (40) with $u(|u|^2-1)^+\varphi_n$ and integrating by parts, we get, as $n \to \infty$,

$$\int_{P\cap[|u|\ge 1]} |\nabla u|^2(|u|^2-1) + \int_{P\cap[|u|\ge 1]} |\nabla |u|^2|^2 \le -\int_{P\cap[|u|\ge 1]} |u|^2(|u|^2-1)^2\,,$$

that is $|u| \le 1$ a.e. It follows that $u \in L^\infty$, so that u may be supposed smooth. $\square$

Proof of Proposition 1. Let $(\varphi_n)_{n\geq 0}$ be an orthonormal basis of eigenfunctions of $-\Delta$ in $H^1_p(P)$ (here "p" means periodic conditions on ∂P) with corresponding eigenvalues $(\lambda_n)_{n\geq 0}$. We may suppose $\varphi_0 = 1$, so that $\lambda_n > 0$ for all $n \geq 1$. If $|\omega_1|, |\omega_2|$ are small enough, then $\lambda_n > 2$ if $n \geq 1$.

Let u be a solution of (40)-(41) and write

$$u = \sum c_n \varphi_n \ , \quad u|u|^2 = \sum d_n \varphi_n \, .$$

Integrating (40) over P, we find that $c_0 = d_0$. Multiplying (40) by φ_n, $n \geq 1$ and integrating we obtain, if $d_n \neq 0$,

$$|d_n| = (\lambda_n - 1)|c_n| > |c_n| \, .$$

Since $|u| \leq 1$, we have

$$\int_P |u|^2 \geq \int_P |u|^6 \, ,$$

that is

$$\sum |c_n|^2 \geq \sum |d_n|^2 \, .$$

Examinating these formulae, we see that $c_n = d_n = 0$ if $n \geq 1$, that is u is constant.□

Concerning the existence of solutions of (40)-(42), we have been able to prove it if P is a rectangle large enough.

Proposition 2. *Let P be large enough such that the first eigenvalue of $-\Delta$ in $H^1_0(R)$ is inferior to 1, where $R = \frac{1}{2}P$.*

Then (40)-(42) has solutions.

Proof. Let

$$J : H^1_0(R) \to \mathbf{R}, \quad J(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int (1 - |u|^2)^2$$

Then J is a C^1-function (see [BN]), even, bounded from below. It is not difficult to see that it satisfies the (PS)-condition:

(PS) if $(u_n) \subset H^1_0(R)$ is such that $(J(u_n))$ is bounded and $J'(u_n) \to 0$ in $H^{-1}(R)$, then (u_n) is relatively compact in $H^1_0(R)$.

Now $J(0) = \dfrac{|R|}{4}$ and, if φ_1 is the first eigenfunction of $-\Delta$ in $H^1_0(R)$, then $J(\varepsilon\varphi_1) < J(0)$ for small ε.

More generally, if the k-th eigenvalue is inferior to 1, one can easily see that there is some $R > 0$ such that $J(u) < J(0)$ if $u \in \mathrm{Sp}\{\varphi_1, ..., \varphi_k\}$ and $\|u\| = R$. Here φ_j denotes the eigenfunction corresponding to the k-th eigenvalue.

It follows from Theorem 8.10 in [R] that J has at least k pairs $(u_j, -u_j)$ of critical points which are different from 0. Let u_0 be a critical point of J in R. Suppose $R = (0\,, a) \times (0\,, b)$.Define $u : P \to \mathbf{C}$ by

$$u(x') = u(x'') = -u_0(x), \quad u(x''') = u_0(x)\,,$$

where $x = (x_1\,, x_2)\,, x' = (2a - x_1\,, x_2)\,, x'' = (x_1\,, 2b - x_2)\,, x''' = (2a - x_1\,, 2b - x_2)$.

It is obvious that u satisfies (41). It is not hard to see that u_0 is regular (see [G]). It follows then by a simple calculation that u satisfies (40).

Finally, suppose (42) does not hold. Let $\vec{\beta} = (\alpha_2, -\alpha_1)$ where $\alpha = \alpha_1 + i\alpha_2$ is as in (42). Then u must be constant along each parallel to $\vec{\beta}$. Since any such line intersects the grid generated by P, it follows that $u \equiv 0$, which is not the case. □

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ON THE GINZBURG-LANDAU ENERGY WITH WEIGHT

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Abstract. We study the behavior as $\varepsilon \to 0$ of minimizers (u_ε) of the Ginzburg-Landau energy E_ε^w with the weight w. We prove the convergence (up to a subsequence) to a harmonic map whose singularities have degree $+1$. We also find the expression of the renormalized energy and deduce that the configuration of singularities is a minimum point of this functional. Our work is motivated by a problem raised by F. Bethuel, H. Brezis and F. Hélein in [4].

Sur l'énergie de Ginzburg-Landau avec poids

Résumé. On étudie le comportement quand $\varepsilon \to 0$ des minimiseurs (u_ε) de l'énergie de Ginzburg-Landau E_ε^w avec le poids w. On montre la convergence (à une sous-suite près) vers une application harmonique dont les singularités ont les degrés $+1$. On trouve aussi l'expression de l'énergie renormalisée et on déduit que la configuration des singularités et un point de minimum de cette fonctionnelle. Notre travail est motivé par un problème posé par F. Bethuel, H. Brezis et F. Hélein dans [4].

Version française abrégée. Soit G un ouvert borné, régulier et simplement connexe dans $\mathbf{R}^2$. On fixe une condition aux limites $g : \partial G \to S^1$ telle que $d = \deg(g, \partial G) > 0$. Soit $w \in C^1(\overline{G}, \mathbf{R})$, $w > 0$ dans $\overline{G}$. On considère l'énergie de Ginzburg-Landau avec le poids w:

$$E_\varepsilon^w(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (1-|u|^2)^2 w\ , \quad \varepsilon > 0\ ,$$

définie pour tout $u \in H^1(G; \mathbf{R}^2)$. Soit u_ε un minimiseur de E_ε^w dans la classe

$$H_g^1(G; \mathbf{R}^2) = \{u \in H^1(G; \mathbf{R}^2);\ u = g \text{ sur } \partial G\}\ .$$

Pour caractériser le comportement des minimiseurs dans le cas $w \equiv 1$, ainsi que la configuration limite, F. Bethuel, H. Brezis et F. Hélein ont défini (voir [2],[4]) l'énergie renormalisée par

$$W(b,\overline{d},g) = -\pi \sum_{i\neq j} d_i d_j \,\log \mid b_i - b_j \mid + \frac{1}{2}\int_{\partial G} \Phi_0 (g \wedge g_\tau) - \pi \sum_{j=1}^{k} d_j R_0(b_j) \;,$$

où $b = (b_1, \cdots, b_k)$ est une configuration de k points distincts dans G de degrés $\overline{d} = (d_1, \cdots, d_k)$, avec $d = d_1 + \cdots + d_k$. Les applications Φ_0 et R_0 sont définies de manière unique par

$$(1) \qquad \begin{cases} \Delta \Phi_0 = 2\pi \displaystyle\sum_{j=1}^{k} d_j \delta_{b_j} \;, & \text{dans } G \\ \dfrac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau \;, & \text{sur } \partial G \\ \displaystyle\int_{\partial G} \Phi_0 = 0 & \end{cases}$$

et

$$(2) \qquad R_0(x) = \Phi_0(x) - \sum_{j=1}^{k} d_j \,\log \mid x - b_j \mid \;.$$

On désigne par $W(b)$ l'énergie renormalisée quand tous les degrés sont égaux à $+1$.

Théorème 1. *Il existe une suite $\varepsilon_n \to 0$ et exactement d points $a_1, \cdots, a_d$ dans G tels que*

$$u_{\varepsilon_n} \to u_\star \quad \textit{dans } H^1_{\text{loc}}(\overline{G} \setminus \{a_1, ..., a_d\}; \mathbf{R}^2)\,,$$

où $u_\star$ est l'application harmonique canonique associée aux singularités $a_1, \cdots, a_d$ de degrés $+1$ et à la donnée au bord g.

De plus, $a = (a_1, \cdots, a_d)$ minimise la fonctionnelle

$$\widetilde{W}(b) = W(b) + \frac{\pi}{2} \sum_{j=1}^{d} \log w(b_j)$$

parmi toutes les configurations $b = (b_1, \cdots, b_d)$ de d points distincts dans G.

On a

$$\lim_{n\to\infty} \{E^w_{\varepsilon_n}(u_{\varepsilon_n}) - \pi d \mid \log \varepsilon_n \mid\} = W(a) + \frac{\pi}{2} \sum_{j=1}^{d} \log w(a_j) + d\gamma\,,$$

où γ est une constante universelle.

Théorème 2. *Soit*

$$W_n = \frac{1}{4\varepsilon_n^2}\,(1- \mid u_{\varepsilon_n} \mid^2)^2 w\,.$$

Alors la suite (W_n) converge dans la topologie faible $\star$ de $C(\overline{G})$ vers

$$W_\star = \frac{\pi}{2} \sum_{j=1}^{d} \delta_{a_j}\,.$$

Let G be a smooth, simply connected domain in $\mathbf{R}^2$ and $w \in C^1(\overline{G}, \mathbf{R})$, $w > 0$ in $\overline{G}$. We consider the Ginzburg-Landau energy with the weight w

$$E_\varepsilon^w(u) = \frac{1}{2}\int_G \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2}\int_G (1- \mid u \mid^2)^2 w\ ,$$

where:

a) $\varepsilon > 0$ is a (small) parameter.

b) $g : \partial G \to S^1$ is a smooth data with a topological degree $d > 0$.

Studying the behavior of minimizers u_ε of E_ε^w in the case $w \equiv 1$, F. Bethuel, H. Brezis and F. Hélein have proved (see [2], [4]) that there exists d points $a_1, \cdots, a_d$ in G such that (up to a subsequence) $u_{\varepsilon_n} \to u_\star$ in $C^k_{\text{loc}}(\overline{G} \setminus \{a_1, \cdots, a_d\})$, where $u_\star$ is the canonical harmonic map associated to g and $a = (a_1, \cdots, a_d)$. In order to locate the singularities at the limit, they have defined the renormalized energy associated to a given configuration $b = (b_1, \cdots, b_k)$ of distinct points in G with associated degrees $\overline{d} = (d_1, \cdots, d_k)$, $d_1 + \cdots + d_k = d$ by

$$W(b, \overline{d}, g) = -\pi \sum_{i \neq j} d_i d_j \ \log \mid b_i - b_j \mid + \frac{1}{2}\int_{\partial G} \Phi_0 (g \wedge g_\tau) - \pi \sum_{j=1}^{k} d_j R_0(b_j)\ ,$$

where Φ_0 is the unique solution of

$$(1) \qquad \begin{cases} \Delta \Phi_0 = 2\pi \displaystyle\sum_{j=1}^{k} d_j \delta_{b_j}\ , & \text{in } G \\ \dfrac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau\ , & \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_0 = 0 & \end{cases}$$

and

$$R_0(x) = \Phi_0(x) - \sum_{j=1}^{k} d_j \log | x - b_j | \ . \tag{2}$$

We shall denote by $W(a)$ the renormalized energy when $k = d$ and all degrees equal +1. F. Bethuel, H. Brezis and F. Hélein have proved in [4] that the functional W is related to the asymptotic behavior of minimizers u_ε as follows:

$$\lim_{\varepsilon \to 0} \{E_\varepsilon(u_\varepsilon) - \pi d \mid \log \varepsilon \mid\} = W(a, \overline{d}, g) + d\gamma \, , \tag{3}$$

where γ is an universal constant, $d_i = 1$ for all i and the configuration $(a_1, \cdots, a_d)$ achieves the minimum of W.

This work is motivated by the Open Problem 2, p. 137 in [4]. We are concerned with the study of the convergence of minimizers of E_ε^w, as well as with the corresponding expression of the renormalized energy. We prove that the behavior of minimizers is of the same type as in the case $w \equiv 1$, the change appearing in the expression of the renormalized energy and, consequently, in the location of singularities of the limit $u_\star$ of u_ε. Our Theorem 2 generalizes another result of F. Bethuel, H. Brezis and F. Hélein concerning the behavior of u_ε. We then prove in Theorem 3 a vanishing gradient property for the configuration of singularities obtained at the limit. The last theorem is devoted to a description of the renormalized energy by the "shrinking holes" method which was developed in [4], Chapter I.

Theorem 1. *There is a sequence $\varepsilon_n \to 0$ and exactly d points $a_1, ..., a_d$ in G such that*

$$u_{\varepsilon_n} \to u_\star \quad \text{in } H^1_{\text{loc}}(\overline{G} \setminus \{a_1, ..., a_d\}; \mathbf{R}^2) \, ,$$

where $u_\star$ is the canonical harmonic map associated to the singularities $a_1, ..., a_d$ of degrees +1 and to the boundary data g.

Moreover, $a = (a_1, ..., a_d)$ minimizes the functional

$$\widetilde{W}(b) = W(b) + \frac{\pi}{2} \sum_{j=1}^{d} \log w(b_j)$$

among all configurations $b = (b_1, ..., b_d)$ of d distinct points in G.

In addition we have

$$\lim_{n \to \infty} \{E_{\varepsilon_n}^w(u_{\varepsilon_n}) - \pi d \mid \log \varepsilon_n \mid\} = W(a) + \frac{\pi}{2} \sum_{j=1}^{d} \log w(a_j) + d\gamma \, ,$$

where γ is some universal constant, the same as in (3).

Remark. The functional $\widetilde{W}$ may be regarded as the renormalized energy corresponding to the energy E_ε^w.

If $c, \varepsilon, \eta > 0$ are constant, let

$$I(\varepsilon,\eta) = \min\{E_\varepsilon(u);\ u \in H^1(B_\eta(0);\mathbf{R}^2) \ \text{and}\ u(x) = \frac{x}{\eta} \ \text{on}\ \partial B_\eta(0)\}\,.$$

For $x \in G$, denote

$$M_\eta(x) = \sup_{B(x,\eta)\cap\overline{G}} w \qquad \text{and} \qquad m_\eta(x) = \inf_{B(x,\eta)\cap\overline{G}} w\ .$$

Sketch of the proof. The first part of the conclusion may be obtained by adapting the techniques developed in [1], [2], [3], [4] taking into account the estimate

$$\frac{1}{\varepsilon_n^2}\int_G (1-\mid u_{\varepsilon_n}\mid^2)^2 w \le C\,, \tag{4}$$

which is deduced by using the ideas in [6].

The proof of the second part of the theorem is divided into 3 steps:

Step 1. *An upper bound for $E_\varepsilon^w(u_\varepsilon)$.* If $b = (b_j)$ is an arbitrary configuration of d distinct points in G, then there exists $\eta_0 > 0$ such that, for each $\eta < \eta_0$,

$$E_\varepsilon^w(u_\varepsilon) \le \sum_{j=1}^d I(\frac{\varepsilon}{\eta\sqrt{M_\eta(b_j)}},1) + W(b) + \pi d \log\frac{1}{\eta} + O(\eta) \quad \text{as}\ \eta \to 0\,, \tag{5}$$

for $\varepsilon > 0$ small enough. Here $O(\eta)$ is a quantity which is bounded by $C\eta$, with C independent of $\eta > 0$ small enough.

Step 2. *A lower bound for $E_{\varepsilon_n}^w(u_{\varepsilon_n})$.* If $a_1, ..., a_d$ are the singularities of $u_\star$ and $\eta > 0$, then there is $N_0 = N_0(\eta) \in$ such that, for each $n \ge N_0$,

$$E_{\varepsilon_n}^w(u_{\varepsilon_n}) \ge \sum_{j=1}^d I\left(\frac{\varepsilon_n}{\alpha\eta\sqrt{m_{\alpha\eta}(a_j)}},1\right) + \pi d\ \log\frac{1}{\eta} + W(a) + O(\eta)\,. \tag{6}$$

Here $\alpha = 1+\eta$ and $O(\eta)$ is a quantity with the same behavior as in (5).

Step 3. *The final conclusion.* From (5), (6) and the asymptotic expression of $I(\varepsilon,\eta)$ as $\frac{\varepsilon}{\eta} \to 0$ (see [4]), we obtain

$$W(b) + \frac{\pi}{2}\sum_{j=1}^d \log M_\eta(b_j) - \pi d \log \varepsilon_n + d\gamma + o(1) \ge \tag{7}$$

$$\geq W(a) + \frac{\pi}{2} \sum_{i=1}^{d} \log m_\eta(a_i) - \pi d \log \varepsilon_n + \pi d \, \log \frac{1}{\eta} - \pi d \, \log \frac{1}{\eta} + d\gamma + o(1) \, ,$$

where $o(1)$ stands for a quantity which goes to 0 as $\varepsilon_n \to 0$ for fixed η. Adding $\pi d \, \log \varepsilon_n$ and passing to the limit firstly as $n \to \infty$ and then as $\eta \to 0$, we obtain that $a = (a_1, ..., a_d)$ is a global minimum point of $\widetilde{W}$. We also deduce that

$$\lim_{n\to\infty} \{E^w_{\varepsilon_n}(u_{\varepsilon_n}) - \pi d \mid \log \varepsilon_n \mid\} = W(a) + \frac{\pi}{2} \sum_{j=1}^{d} \log w(a_j) + d\gamma \ .$$

Theorem 2. *Set*

$$W_n = \frac{1}{4\varepsilon_n^2} (1- \mid u_{\varepsilon_n} \mid^2)^2 w \, .$$

Then (W_n) *converges in the weak* $\star$ *topology of* $C(\overline{G})$ *to*

$$W_\star = \frac{\pi}{2} \sum_{j=1}^{d} \delta_{a_j} \ .$$

The expression of the renormalized energy $\widetilde{W}$ allows us, by using the results obtained in [4], to give the analogue of the vanishing gradient property obtained in [4], Chapter VIII.2.

Taking into account Theorem 1 and using the expression of DW (see Theorem VIII.3 in [4]) we obtain

Theorem 3. ("Vanishing gradient property") *If* $a = (a_1, \cdots, a_d)$ *is as in Theorem 1, then*

$$\nabla R_0(a_j) + \sum_{i\neq j} \frac{a_j - a_i}{\mid a_j - a_i \mid^2} = \frac{1}{4} \frac{\nabla w(a_j)}{w(a_j)}, \quad \textit{for each} \ \ j \, .$$

As in [4], Chapter I.4, we may define the renormalized energy by considering a suitable variational problem in a domain with shrinking holes.

Let $b_1, ..., b_k$ be distinct points in G. Fix $d_1, ..., d_k \in$ and a smooth data $g : \partial G \to S^1$ of degree $d = d_1 + ... + d_k$. For each $\eta > 0$ small enough, define

$$G^w_\eta = G \setminus \bigcup_{j=1}^{k} \overline{\omega_{j,\eta}} \, ,$$

where

$$\omega_{j,\eta} = B\left(b_j, \frac{\eta}{\sqrt{w(b_j)}}\right).$$

Set

$$\mathcal{E}_\eta^w = \{v \in H^1(G_\eta^w; S^1)\,;\ \deg(v, \partial\omega_{j,\eta}) = d_j \ \text{ and } \ v = g \ \text{ on } \ \partial G\}.$$

Let u_η be a solution of

$$\min_{u \in \mathcal{E}_\eta^w} \int_{G_\eta^w} |\nabla u|^2 \ . \tag{8}$$

The following result shows that the renormalized energy $\widetilde{W}$ is what remains in the energy after the singular "core energy" $\pi\, d\ |\log\eta|$ has been removed.

Theorem 4. *We have the following asymptotic estimate:*

$$\frac{1}{2}\int_{G_\eta^w} |\nabla u_\eta|^2 = \pi\Big(\sum_{j=1}^k d_j^2\Big)\ |\log\eta| + \widetilde{W}(b,\overline{d},g) + O(\eta), \quad \text{as } \eta \to 0\,,$$

where

$$\widetilde{W}(b,\overline{d},g) = W(b,\overline{d},g) + \frac{\pi}{2}\left(\sum_{j=1}^k d_j^2 \log w(b_j)\right).$$

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ON THE GINZBURG-LANDAU ENERGY WITH WEIGHT

(Sur l'énergie de Ginzburg-Landau avec poids)

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Abstract. This paper gives a solution to an open problem raised by Bethuel, Brezis and Hélein. We study the Ginzburg-Landau energy with weight. We find the expression of the renormalized energy and we show that the finite configuration of singularities of the limit is a minimum point of this functional. We find a vanishing gradient type property and then we obtain the renormalized energy by Bethuel, Brezis and Hélein's shrinking holes method.

Résumé. Ce travail donne la solution d'un problème ouvert de Bethuel, Brezis and Hélein. On étudie l'énergie de Ginzburg-Landau avec poids. Nous trouvons l'expression de l'énergie renormalisée et on prouve que la configuration finie des singularités de la limite est un point de minimum pour cette fonctionelle. Nous montrons une propriété du type "vanishing gradient" et on obtient ensuite l'énergie renormalisée avec la méthode "shrinking holes" de Bethuel, Brezis et Hélein.

Keywords: Ginzburg-Landau energy with weight, renormalized energy.

Classification A.M.S. : 35 J 60, 35 Q 99.

1. Introduction

In a recent book [BBH4], F. Bethuel, H. Brezis and F. Hélein studied the vortices related to the Ginzburg-Landau functional. Similar functionals appear in the study of problems occuring in superconductivity or the theory of superfluids.

In [BBH4], F. Bethuel, H. Brezis and F. Hélein have studied the behavior as $\varepsilon \to 0$ of minimizers u_ε of the Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2}\int_G \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2}\int_G (1- \mid u \mid^2)^2$$

in the class of functions

$$H^1_g(G) = \{u \in H^1(G; \mathbf{R}^2);\ u = g \ \text{ on } \partial G\}\,,$$

where:

a) $\varepsilon > 0$ is a (small) parameter.

b) G is a smooth, simply connected, starshaped domain in $\mathbf{R}^2$.

c) $g : \partial G \to S^1$ is a smooth data with a topological degree $d > 0$.

They obtained the convergence of (u_{ε_n}) in certain topologies to $u_\star$. The function $u_\star$ is a *harmonic map* from $G \setminus \{a_1, ..., a_d\}$ to S^1, and is *canonical,* in the sense that

$$\frac{\partial}{\partial x_1}\left(u_\star \wedge \frac{\partial u_\star}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(u_\star \wedge \frac{\partial u_\star}{\partial x_2}\right) = 0 \quad \text{in} \;\; \mathcal{D}'(G)\,.$$

Recall (see [BBH4]) that a canonical harmonic map $u_\star$ with values in S^1 and singularities $b_1, ..., b_k$ of degrees $d_1, ..., d_k$ may be expressed as

$$u_\star(x) = \left(\frac{x - b_1}{\mid x - b_1 \mid}\right)^{d_1} \cdots \left(\frac{x - b_k}{\mid x - b_k \mid}\right)^{d_k} e^{i\varphi_0(x)}\,,$$

with

$$\Delta\varphi_0 = 0 \qquad \text{in} \;\; G\,.$$

They also defined the notion of renormalized energy $W(b, \overline{d}, g)$ associated to a given configuration $b = (b_1, ..., b_k)$ of distinct points with associated degrees $\overline{d} = (d_1, ..., d_k)$. For simplicity we set $W(b) = W(b, \overline{d}, g)$ when $k = d$ and all the degrees equal $+1$. The expression of the renormalized energy W is given by

$$W(b, \overline{d}, g) = -\pi \sum_{i \neq j} d_i d_j \, \log \mid b_i - b_j \mid + \frac{1}{2}\int_{\partial G} \Phi_0 (g \wedge g_\tau) - \pi \sum_{j=1}^{k} d_j R_0(b_j)\,,$$

where Φ_0 is the unique solution of

$$(1) \qquad \begin{cases} \Delta\Phi_0 = 2\pi \displaystyle\sum_{j=1}^{k} d_j \delta_{b_j}\,, & \text{in } G \\ \dfrac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau\,, & \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_0 = 0 & \end{cases}$$

and

$$R_0(x) = \Phi_0(x) - \sum_{j=1}^{k} d_j \, \log \mid x - b_j \mid \,.$$

The functional W is also related to the asymptotic behavior of minimizers u_ε as follows:

$$\lim_{\varepsilon\to 0}\{E_\varepsilon(u_\varepsilon)-\pi d\mid\log\varepsilon\mid\}=\min_{b\in G^d}W(b)+d\gamma\,, \tag{2}$$

where γ is an universal constant, $k=d$, $d_i=+1$ for all i and the configuration $a=(a_1,\cdots,a_d)$ achieves the minimum of W.

We study in this paper a similar problem, related to the Ginzburg-Landau energy with the weight w, that is

$$E_\varepsilon^w(u)=\frac{1}{2}\int_G\mid\nabla u\mid^2+\frac{1}{4\varepsilon^2}\int_G(1-\mid u\mid^2)^2w\,,$$

with $w\in C^1(\overline{G})$, $w>0$ in $\overline{G}$. Throughout, u_ε will denote a minimizer of E_ε^w. We mention that u_ε verifies the Ginzburg-Landau equation with weight

$$\begin{cases}-\Delta u_\varepsilon=\dfrac{1}{\varepsilon^2}\,u_\varepsilon(1-\mid u_\varepsilon\mid^2)w & \text{in } G\\ u_\varepsilon=g \quad \text{on } \partial G\,.\end{cases} \tag{3}$$

Our work is motivated by the Open Problem 2, p. 137 in [BBH4]. We are concerned in this paper with the study of the convergence of minimizers, as well as with the corresponding expression of the renormalized energy. We prove that the behavior of minimizers is of the same type as in the case $w\equiv 1$, the change appearing in the expression of the renormalized energy and, consequently, in the location of singularities of the limit $u_\star$ of u_{ε_n}. In our proof we borrow some of the ideas from Chapter VIII in [BBH4], without relying on the vanishing gradient property that is used there. We then prove a corresponding vanishing gradient property for the configuration of singularities obtained at the limit. In the last section we obtain the new renormalized energy by a variant of the "shrinking holes" method which was developed in [BBH4], Chapter I.

2. The renormalized energy

Theorem 1. *There is a sequence $\varepsilon_n\to 0$ and exactly d points $a_1,...,a_d$ in G such that*

$$u_{\varepsilon_n}\to u_\star \quad \textit{in } H^1_{\rm loc}(\overline{G}\setminus\{a_1,...,a_d\};\mathbf{R}^2)\,,$$

where $u_\star$ is the canonical harmonic map associated to the singularities $a_1, ..., a_d$ of degrees +1 and to the boundary data g.

Moreover, $a = (a_1, \cdots, a_d)$ minimizes the functional

$$\widetilde{W}(b) = W(b) + \frac{\pi}{2}\sum_{j=1}^{d} \log w(b_j) \tag{4}$$

among all configurations $b = (b_1, ..., b_d)$ of d distinct points in G.

In addition, the following holds:

$$\lim_{n\to\infty} \{E^w_{\varepsilon_n}(u_{\varepsilon_n}) - \pi d \mid \log \varepsilon_n \mid\} = W(a) + \frac{\pi}{2}\sum_{j=1}^{d} \log w(a_j) + d\gamma\,, \tag{5}$$

where γ is some universal constant, the same as in (2).

Remark. The functional $\widetilde{W}$ may be regarded as the renormalized energy corresponding to the energy E^w_ε.

Before giving the proof, we shall make some useful notations: given the constants $c, \varepsilon, \eta > 0$, set

$$I^c(\varepsilon, \eta) = \min\{E^c_\varepsilon(u);\ u \in H^1(B_\eta; \mathbf{R}^2) \text{ and } u(x) = \frac{x}{\eta} \text{ on } \partial B_\eta\}\,.$$

Here $B_\eta = B(0, \eta) \subset \mathbf{R}^2$.

For $x \in G$, denote

$$M_\eta(x) = \sup_{B(x,\eta)\cap\overline{G}} w \qquad \text{and} \qquad m_\eta(x) = \inf_{B(x,\eta)\cap\overline{G}} w\,.$$

Note that

$$I^c(\varepsilon, \eta) = I^c(\frac{\varepsilon}{\eta}, 1) = I^1(\frac{\varepsilon}{\eta\sqrt{c}}, 1)$$

and

$$I^{c_1}(\varepsilon, \eta) \le I^{c_2}(\varepsilon, \eta)\,,$$

provided $c_1 \le c_2$.

We shall drop the superscript c if it equals 1.

Proof of Theorem 1. The first part of the conclusion may be obtained by adapting the techniques developed in [BBH1], [BBH2], [BBH3], [BBH4] (see also [S]). We shall point out only the main steps that are necessary to prove the convergence:

a) Using the techniques from [S] we find a sequence $\varepsilon_n \to 0$ such that, for each n,

$$\frac{1}{\varepsilon_n^2}\int_G (1- \mid u_{\varepsilon_n} \mid^2)^2 w \le C\,. \tag{6}$$

b) Using the methods developed in [BBH4], Chapters 3-5, we determine the "bad" disks, as well as the fact that their number is uniformly bounded. These techniques allow us to prove the convergence of (u_{ε_n}) weakly in $H^1_{\rm loc}(G\setminus\{a_1,...,a_k\};\mathbf{R}^2)$ to $u_\star$, which is the canonical harmonic map associated to $a_1,...,a_k$ with some degrees $d_1,...,d_k$ and to the given boundary data.

c) The strong convergence of (u_{ε_n}) in $H^1_{\rm loc}(\overline{G}\setminus\{a_1,...,a_k\};\mathbf{R}^2)$ follows as in [BBH4], Theorem VI.1 with the techniques from [BBH3], Theorem 2, Step 1. Now the local convergence of (u_{ε_n}) in $G\setminus\{a_1,\cdots,a_k\}$ in stronger topologies, say C^2, may be easily obtained by a bootstrap argument in (3). This implies that

$$\frac{1- \mid u_{\varepsilon_n}\mid^2}{\varepsilon_n^2} w \to \mid \nabla u_\star \mid^2 , \tag{7}$$

uniformly on every compact subset of $G\setminus\{a_1,...,a_k\}$.

d) For each $1 \le j \le k$, deg $(u_\star, a_j) \neq 0$. Indeed, if not, then as in Step 1 of Theorem 2 [BBH3], the H^1-convergence is extended up to a_j, which becomes a "removable singularity".

e) The fact that all degrees equal +1 may be deduced as in Theorem VI.2, [BBH4].

f) The points $a_1,...,a_d$ lie in G. The proof of this fact is similar to the corresponding result in [BBH4].

The proof of the second part of the theorem is divided into 3 steps:

Step 1. *An upper bound for $E^w_\varepsilon(u_\varepsilon)$.*

We shall prove that if $b=(b_j)$ is an arbitrary configuration of d distinct points in G, then there exists $\eta_0>0$ such that, for each $\eta<\eta_0$,

$$E^w_\varepsilon(u_\varepsilon) \le \sum_{j=1}^d I(\frac{\varepsilon}{\eta\sqrt{M_\eta(b_j)}},1) + W(b) + \pi d \log\frac{1}{\eta} + O(\eta) \quad \text{as } \eta \to 0\,, \tag{8}$$

for $\varepsilon>0$ small enough. Here $O(\eta)$ is a quantity which is bounded by $C\eta$, with C independent of $\eta>0$ small enough.

The idea is to construct a suitable comparison function v_ε. Let $\eta<\eta_0$, where $\eta_0 = \min\limits_{j,k}\{\text{dist}\,(b_j,\partial G), \mid b_j - b_k \mid\}$. Applying Theorem I.9 in [BBH4] to the configuration

b, we find $\widetilde{u} : G_\eta := G \setminus \bigcup_{j=1}^{d} \overline{B(b_j,\eta)} \to S^1$ with $\widetilde{u} = g$ on ∂G and $\alpha_j \in$, $| \alpha_j | = 1$ such that

$$\widetilde{u} = \alpha_j \frac{z - b_j}{| z - b_j |} \quad \text{on} \quad \partial B(b_j, \eta)$$

and

$$\frac{1}{2} \int_{G_\eta} | \nabla \widetilde{u} |^2 = \pi d \log \frac{1}{\eta} + W(b) + O(\eta)\,, \quad \text{as } \eta \to 0\,. \tag{9}$$

We define v_ε as follows: let $v_\varepsilon = \widetilde{u}$ on G_η and, in $B(b_j, \eta)$, let v_ε be a minimizer of E^w_ε on $H^1_h(B(b_j,\eta); \mathbf{R}^2)$, where $h = \widetilde{u} \,|_{\partial B(b_j,\eta)}$. We have the following estimate

$$E^w_\varepsilon (v_\varepsilon \,|_{B(b_j,\eta)}) \le I^{M_\eta(b_j)}(\varepsilon, \eta) = I(\frac{\varepsilon}{\eta \sqrt{M_\eta(b_j)}}, 1)\,. \tag{10}$$

The desired conclusion follows from (9),(10) and $E^w_\varepsilon(u_\varepsilon) \le E^w_\varepsilon(v_\varepsilon)$.

Step 2. *A lower bound for $E^w_{\varepsilon_n}(u_{\varepsilon_n})$.*

We shall prove that, if $a_1, ..., a_d$ are the singularities of $u_\star$, then given any $\eta > 0$, there is $N_0 = N_0(\eta) \in$ such that, for each $n \ge N_0$,

$$E^w_{\varepsilon_n}(u_{\varepsilon_n}) \ge \sum_{j=1}^{d} I\left(\frac{\varepsilon_n}{\alpha\eta\sqrt{m_{\alpha\eta}(a_j)}}, 1 \right) + \pi d \,\log \frac{1}{\eta} + W(a) + O(\eta)\,. \tag{11}$$

Here $\alpha = 1 + \eta$ and $O(\eta)$ is a quantity with the same behavior as in (8).

Indeed, for a fixed a_j, supposed to be 0, $u_\star$ may be written

$$u_\star = e^{i(\psi + \theta)}\,,$$

where ψ is a smooth harmonic function in a neighbourhood of 0. We may assume, without loss of generality, that $\psi(0) = 0$.

In the annulus $A_{\eta,\alpha\eta} = \{x \in \mathbf{R}^2 \;;\; \eta \le | x | \le \alpha\eta\}$ the function u_{ε_n} may be written, for n large enough, as

$$u_{\varepsilon_n} = \rho_n e^{i(\psi_n + \theta)}\,,$$

where ψ_n is a smooth function and $0 < \rho_n \le 1$. Define, for $\eta \le r \le \alpha\eta$, the interpolation function

$$v_n(r,\theta) = \frac{r - \eta + \rho_n(\eta,\theta)(\alpha\eta - r)}{\eta(\alpha - 1)} \cdot e^{i[\frac{\alpha\eta - r}{\eta(\alpha-1)}\psi_n(\eta,\theta) + \theta]}\,.$$

We have

$$\frac{1}{\varepsilon_n^2}\int_{A_{\eta,\alpha\eta}}(1-\mid v_n\mid^2)^2 w\le\frac{\|w\|_{L^\infty}}{\varepsilon_n^2}\cdot\int_\eta^{\alpha\eta}\frac{r}{\eta}(\int_{\partial B_\eta}(1-\mid u_n\mid^2)^2 d\sigma)dr=$$

$$=\|w\|_{L^\infty}\cdot\frac{\alpha+1}{2}\eta^2\int_{\partial B_\eta}\frac{(1-\mid u_n\mid^2)^2}{\varepsilon_n^2}d\sigma\to 0\,,\quad \text{as } n\to\infty\,.$$

This convergence is motivated by (7). We also observe that the convergence of (u_{ε_n}) in $H^1_{\rm loc}(G\setminus\{a_1,...,a_d\};\mathbf{R}^2)$ implies

(12) $$\int_{A_{\eta,\alpha\eta}}\mid\nabla v_n\mid^2\to\int_{A_{\eta,\alpha\eta}}\mid\nabla v\mid^2\,,\quad \text{as}\ \ \eta\to 0\,,$$

where

$$v(\eta,\theta)=e^{i[\frac{\alpha\eta-r}{\eta(\alpha-1)}\psi(\eta,\theta)+\theta]}\,.$$

Thus, we may write, for $n\ge N_1$,

$$E^w_{\varepsilon_n}(v_n\mid_{A_{\eta,\alpha\eta}})=\frac{1}{2}\int_{A_{\eta,\alpha\eta}}\mid\nabla v\mid^2+o(1)\,.$$

We prove in what follows that

(13) $$\int_{A_{\eta,\alpha\eta}}\mid\nabla v\mid^2=O(\eta)\,.$$

Indeed, since

$$\mid\nabla v\mid^2=\frac{\psi^2(\eta,\theta)}{\eta^2(\alpha-1)^2}+\frac{1}{r^2}\left[\frac{\alpha\eta-r}{\eta(\alpha-1)}\psi_\theta(\eta,\theta)+1\right]^2$$

and

$$\psi(r,\theta)\le Cr\,,\quad\mid\psi_r(r,\theta)\mid\le C\,,\quad\mid\psi_\theta(r,\theta)\mid\le Cr\,,$$

the desired conclusion follows by a straightforward calculation.

We obtain

(14) $$E^w_{\varepsilon_n}(v_{\varepsilon_n|_{B_{(a_j,\eta)}}})\ge I^{m_{\alpha\eta}(a_j)}(\varepsilon_n,\alpha\eta)+O(\eta)\,.$$

On the other hand, by the convergence of (u_{ε_n}) in $H^1_{\rm loc}(G\setminus\{a_1,...,a_d\};\mathbf{R}^2)$ it follows that

(15) $$E^w_{\varepsilon_n}(u_{\varepsilon_n}\mid_{G_\eta})=\int_{G_\eta}\mid\nabla u_\star\mid^2+O(\eta)\,,$$

for ε_n sufficiently small.

Taking into account (12)-(15) we obtain the desired result.

Step 3. *The final conclusion.*

It follows from [BBH4], Chapter IX that

$$I(\varepsilon,\eta)=\pi\mid\log\frac{\varepsilon}{\eta}\mid+\gamma+o(1)\quad \text{as}\quad \frac{\varepsilon}{\eta}\to 0\,, \tag{16}$$

where the constant γ represents the minimum of the renormalized energy corresponding to the boundary data x in B_1.

From (8) and (11) we obtain

$$W(b)+\frac{\pi}{2}\sum_{j=1}^{d}\log M_\eta(b_j)-\pi d\log\varepsilon_n+d\gamma+o(1)\geq \tag{17}$$

$$\geq W(a)+\frac{\pi}{2}\sum_{i=1}^{d}\log m_\eta(a_i)-\pi d\log\varepsilon_n+\pi d\,\log\frac{1}{\eta}-\pi d\,\log\frac{1}{\eta}+d\gamma+o(1)\,,$$

where $o(1)$ stands for a quantity which goes to 0 as $\varepsilon_n\to 0$ for fixed η. Adding $\pi d\,\log\varepsilon_n$ and passing to the limit firstly as $n\to\infty$ and then as $\eta\to 0$, we obtain that $a=(a_1,...,a_d)$ is a global minimum point of $\widetilde{W}$. We also deduce that

$$\lim_{n\to\infty}\{E^w_{\varepsilon_n}(u_{\varepsilon_n})-\pi d\mid\log\varepsilon_n\mid\}=W(a)+\frac{\pi}{2}\sum_{j=1}^{d}\log w(a_j)+d\gamma\ .$$

We now generalize another result from [BBH4] concerning the behavior of u_ε.

Theorem 2. *Set*

$$W_n=\frac{1}{4\varepsilon_n^2}\,(1-\mid u_{\varepsilon_n}\mid^2)^2w\,.$$

Then (W_n) *converges in the weak* $\star$ *topology of* $C(\overline{G})$ *to*

$$W_\star=\frac{\pi}{2}\sum_{j=1}^{d}\delta_{a_j}\,.$$

Proof. The boundedness of (W_n) in $L^1(G)$ follows directly from (6). Hence (up to a subsequence), W_n converges in the sense of measures of $\overline{G}$ to some $W_\star$. With the same techniques as those developed in [BBH3] (Theorem 2) or [BBH4] (Theorem X.3) we can obtain that, for any compact subset K of $\overline{G} \setminus \bigcup_{j=1}^{d} \{a_j\}$,

$$\frac{1}{\varepsilon_n^2} \|1- \mid u_{\varepsilon_n} \mid^2 \|_{L^\infty(K)} \leq C_K \ .$$

Hence

$$\text{supp } W_\star \subset \bigcup_{j=1}^{d} \{a_j\} \,.$$

Therefore

$$W_\star = \sum_{j=1}^{d} m_j \delta_{a_j} \qquad \text{with } \ m_j \in \mathbf{R} \,.$$

We now determine m_j using the same methods as in [BBH4]. Fix one of the points a_j (supposed to be 0) and consider $B_R = B(0, R)$ for R small enough so that B_R contains no other point a_i $(i \neq j)$. As in the proof of the Pohozaev identity, multiplying the Ginzburg-Landau equation (3) by $x \cdot \nabla u_\varepsilon$ and integrating on B_R we obtain

$$\frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\varepsilon}{\partial \nu} \mid^2 + \frac{1}{2\varepsilon^2} \int_{B_R} (1- \mid u_\varepsilon \mid^2)^2 w + \frac{1}{4\varepsilon^2} \int_{B_R} (1- \mid u_\varepsilon \mid^2)^2 (\nabla w \cdot x) = \tag{18}$$

$$= \frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\varepsilon}{\partial \tau} \mid^2 + \frac{R}{4\varepsilon^2} \int_{\partial B_R} (1- \mid u_\varepsilon \mid^2)^2 w \,.$$

Passing to the limit in (18) as $\varepsilon \to 0$ and using the convergence of W_n we find

$$\frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\star}{\partial \nu} \mid^2 + 2m_j = \frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\star}{\partial \tau} \mid^2 \ . \tag{19}$$

Using now the expression of $u_\star$ around a singularity we deduce that, on ∂B_R,

$$\mid \frac{\partial u_\star}{\partial \nu} \mid^2 = \mid \frac{\partial \theta}{\partial \nu} + \frac{\partial \psi}{\partial \nu} \mid^2 = \mid \frac{\partial \psi}{\partial \nu} \mid^2 \ . \tag{20}$$

$$\mid \frac{\partial u_\star}{\partial \tau} \mid^2 = \mid \frac{\partial \theta}{\partial \tau} + \frac{\partial \psi}{\partial \tau} \mid^2 = \frac{1}{R^2} + \frac{2}{R} \frac{\partial \psi}{\partial \tau} + \mid \frac{\partial \psi}{\partial \tau} \mid^2 \ . \tag{21}$$

Inserting (20) and (21) into (19) we obtain

$$\frac{R}{2} \int_{\partial B_R} \mid \frac{\partial \psi}{\partial \nu} \mid^2 + 2m_j = \pi + \frac{R}{2} \int_{\partial B_R} \mid \frac{\partial \psi}{\partial \tau} \mid^2 \ . \tag{22}$$

On the other hand, multiplying $\Delta\psi = 0$ by $x \cdot \nabla\psi$ and integrating on B_R we find

$$\frac{R}{2}\int_{\partial B_R} \mid \frac{\partial\psi}{\partial\nu} \mid^2 = \frac{R}{2}\int_{\partial B_R} \mid \frac{\partial\psi}{\partial\tau} \mid^2 \quad . \tag{23}$$

Thus, from (17) and (18) we obtain

$$m_j = \frac{\pi}{2} \ .$$

□

3. The vanishing gradient property of the renormalized energy with weight

The expression of the renormalized energy $\widetilde{W}$ allows us, by using the results obtained in [BBH4], to give an expression of the vanishing gradient property in the case of a weight.

From (4) it follows that

$$D\widetilde{W}(b_1, ..., b_d) = DW(b_1, ..., b_d) + \frac{\pi}{2}\left(\frac{\nabla w(b_1)}{w(b_1)}, ..., \frac{\nabla w(b_d)}{w(b_d)}\right), \tag{24}$$

for each configuration $b = (b_1, ..., b_d) \in G^d$.

Recall now Theorem VIII.3 in [BBH4], which gives the expression of the differential of W in an arbitrary configuration of distinct points $b = (b_1, ..., b_d) \in G^d$:

$$DW(b) = -2\pi\left[\left(\frac{\partial S_1}{\partial x_1}(b_1), \frac{\partial S_1}{\partial x_2}(b_1)\right), ..., \left(\frac{\partial S_d}{\partial x_1}(b_d), \frac{\partial S_d}{\partial x_2}(b_d)\right)\right] = \tag{25}$$

$$= 2\pi\left[\left(-\frac{\partial H_1}{\partial x_2}(b_1), \frac{\partial H_1}{\partial x_1}(b_1)\right), ..., \left(-\frac{\partial H_d}{\partial x_2}(b_d), \frac{\partial H_d}{\partial x_1}(b_d)\right)\right].$$

Here $S_j(x) = \Phi_0(x) - \log \mid x - b_j \mid$ in G and Φ_0 the unique solution of

$$\begin{cases} \Delta\Phi_0 = 2\pi \sum_{j=1}^{d} \delta_{b_j} \text{ , in } G \\ \dfrac{\partial\Phi_0}{\partial\nu} = g \wedge g_\tau \text{ , on } \partial G \\ \displaystyle\int_{\partial G} \Phi_0 = 0. \end{cases}$$

The function H_j is harmonic around b_j and is related to $u_\star$ by

$$u_\star(x) = \frac{x - b_j}{| x - b_j |} e^{iH_j(x)} \ , \quad \text{near } b_j \ .$$

Let

$$R_0(x) = S_j(x) - \sum_{i \neq j} \log | x - b_i | \ .$$

Our variant of the vanishing gradient property in [BBH4] (Corollary VIII.1) is:

Theorem 3. *The following properties are equivalent:*
i) $a = (a_1, ..., a_d)$ *is a critical point of the renormalized energy* $\widetilde{W}$.
ii) $\nabla S_j(a_j) = \frac{1}{4} \frac{\nabla w(a_j)}{w(a_j)}$, *for each* j.
iii) $\nabla H_j(a_j) = \frac{1}{4w(a_j)} \left(-\frac{\partial w}{\partial x_2}(a_j), \frac{\partial w}{\partial x_1}(a_j) \right)$, *for each* j.
iv) $\nabla R_0(a_j) + \sum_{i \neq j} \frac{a_j - a_i}{| a_j - a_i |^2} = \frac{1}{4} \frac{\nabla w(a_j)}{w(a_j)}$, *for each* j.

The proof follows by the above considerations and the fact that, for each j,

$$\nabla R_0(x) = \nabla S_j(x) - \sum_{i \neq j} \frac{x - a_i}{| x - a_i |^2} \ .$$

4. Shrinking holes and the renormalized energy with weight

As in [BBH4], Chapter I.4, we may define the renormalized energy by considering a suitable variational problem in a domain with "shrinking holes".

Let, as above, G be a smooth, bounded and simply connected domain in $\mathbf{R}^2$ and let $b_1, ..., b_k$ be distinct points in G. Fix $d_1, ..., d_k \in$ and a smooth data $g : \partial G \to S^1$ of degree $d = d_1 + ... + d_k$. For each $\eta > 0$ small enough, define

$$G_\eta^w = G \setminus \bigcup_{j=1}^{k} \overline{\omega_{j,\eta}} \, ,$$

where

$$\omega_{j,\eta} = B\left(b_j, \frac{\eta}{\sqrt{w(b_j)}} \right) .$$

Set

$$\mathcal{E}_\eta^w = \{v \in H^1(G_\eta^w; S^1)\,;\ \deg(v, \partial\omega_{j,\eta}) = d_j \ \text{ and } \ v = g \ \text{ on } \ \partial G\}\,.$$

We consider the minimization problem

$$\min_{u \in \mathcal{E}_\eta^w} \int_{G_\eta^w} |\nabla u|^2 \ . \tag{26}$$

The following result shows that the renormalized energy $\widetilde{W}$ is what remains in the energy after the singular "core energy" $\pi\, d\ |\log \eta|$ has been removed.

Theorem 4. *We have the following asymptotic estimate:*

$$\frac{1}{2}\int_{G_\eta^w} |\nabla u_\eta|^2 = \pi\Big(\sum_{j=1}^k d_j^2\Big)\,|\log\eta| + \widetilde{W}(b,\overline{d},g) + O(\eta), \quad \textit{as } \eta \to 0\,,$$

where

$$\widetilde{W}(b,\overline{d},g) = W(b,\overline{d},g) + \frac{\pi}{2}\left(\sum_{j=1}^k d_j^2 \log w(b_j)\right).$$

Proof. As in [BBH4], Chapter I we associate to (26) the linear problem:

$$\begin{cases} \Delta\Phi_\eta = 0\ , \quad \text{in } G_\eta^w \\ \Phi_\eta = C_j = \text{Const.}\ , \quad \text{on each } \partial\omega_{j,\eta} \\ \displaystyle\int_{\partial\omega_{j,\eta}} \frac{\partial\Phi_\eta}{\partial\nu} = 2\pi d_j\ , \quad \text{for each } j = 1,...,k \\ \displaystyle\frac{\partial\Phi_\eta}{\partial\nu} = g \wedge g_\tau\ , \quad \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_\eta = 0. \end{cases} \tag{27}$$

With the same techniques as in [BBH4] (see Lemma I.2), one may prove that

$$\|\Phi_\eta - \Phi_0\|_{L^\infty(G_\eta^w)} = O(\eta)\,,$$

where Φ_0 is the unique solution of (1).

Note that the link between Φ_η and an arbitrary solution u_η of (26) is

$$\begin{cases} u_\eta \wedge \dfrac{\partial u_\eta}{\partial x_1} = -\dfrac{\partial\Phi_\eta}{\partial x_2} \quad \text{in } G_\eta^w \\ u_\eta \wedge \dfrac{\partial u_\eta}{\partial x_2} = \dfrac{\partial\Phi_\eta}{\partial x_1} \quad \text{in } G_\eta^w \end{cases} \tag{28}$$

From now on the proof follows the same lines as of Theorem I.7 in [BBH4]. □

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MINIMIZATION PROBLEMS AND CORRESPONDING RENORMALIZED ENERGIES

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1. Introduction

Let G be a smooth bounded simply connected domain in $\mathbf{R}^2$. Let $a = (a_1, ..., a_k)$ be a configuration of distinct points in G and $\overline{d} = (d_1, ..., d_k) \in \mathbf{Z}^k$. Consider a smooth boundary data $g : \partial G \to S^1$ whose topological degree is $d = d_1 + ... + d_k$. Let also $\rho > 0$ be sufficiently small and denote

$$\Omega_\rho = G \setminus \bigcup_{i=1}^{k} \overline{B(a_i, \rho)} \ , \ \Omega = G \setminus \{a_1, ...a_k\} \ .$$

In [BBH4], B. Bethuel, H. Brezis and F. Hélein have studied the behavior as $\rho \to 0$ of solutions of the minimization problem

$$E_{\rho,g} = \min_{u \in \mathcal{E}_{\rho,g}} \int_{\Omega_\rho} | \nabla v |^2 , \tag{1}$$

where

$$\mathcal{E}_{\rho,g} = \{v \in H^1(\Omega_\rho; S^1); \ v = g \text{ on } \partial G \text{ and } \deg(v, \partial B(a_i, \rho)) = d_i, \text{ for } i = 1, ..., k\} .$$

They proved that (1) has a unique solution, say u_ρ. By analysing the behavior of u_ρ as $\rho \to 0$, they obtained the renormalized energy $W(a, \overline{d}, g)$ through the following asymptotic expansion:

$$\frac{1}{2} \int_{\Omega_\rho} | \nabla u_\rho |^2 = \pi \left(\sum_{i=1}^{k} d_i^2 \right) \log \frac{1}{\rho} + W(a, \overline{d}, g) + O(\rho) \ , \quad \text{as } \rho \to 0. \tag{2}$$

If $G = B_1$ and $g(\theta) = e^{di\theta}$ we give an explicit formula for $W(a, \overline{d}, g)$:

$$W(a, \overline{d}, g) == -\pi \sum_{i \neq j} d_i d_j \log | a_i - a_j | -\pi \sum_{i,j} d_i d_j \log | 1 - a_i \overline{a}_j | \quad . \tag{3}$$

It is natural to ask what happens if we try to minimize the Dirichlet energy $\int_{\Omega_\rho} | \nabla v |^2$ with respect to other classes of test functions. Let

$$\mathcal{F}_\rho = \{v \in H^1(\Omega_\rho; S^1);\ \deg(v, \partial G) = d \text{ and } \deg(v, \partial B(a_i, \rho)) = d_i,\ \text{for } i = 1, ..., k\}\,.$$

In [BBH4] it is proved that the problem

$$F_\rho = \min_{u \in \mathcal{F}_\rho} \int_{\Omega_\rho} | \nabla v |^2\,, \tag{4}$$

has a unique solution v_ρ. We find an analogous asymptotic estimate of (2) for the problem (4). More precisely, we prove that

$$\frac{1}{2} \int_{\Omega_\rho} | \nabla v_\rho |^2 = \pi \left(\sum_{i=1}^k d_i^2 \right) \log \frac{1}{\rho} + \widetilde{W}(a, \overline{d}) + O(\rho)\ , \quad \text{as } \rho \to 0. \tag{5}$$

The connection between the renormalized energy $W(a, \overline{d}, g)$ from [BBH4] and the new renormalized energy $\widetilde{W}(a, \overline{d})$ is

$$\widetilde{W}(a, \overline{d}) = \inf_{\substack{g:\partial G \to S^1 \\ \deg(g, \partial G) = d}} W(a, \overline{d}, g). \tag{6}$$

Moreover the infimum in (6) is achieved. In the case $G = B_1$ we prove that

$$\widetilde{W}(a, \overline{d}) = -\pi \sum_{i \neq j} d_i d_j \log | a_i - a_j | +\pi \sum_{i,j} d_i d_j \log | 1 - a_i \overline{a}_j | \quad . \tag{7}$$

We also study the behavior as $\rho \to 0$ of solutions of the minimization problem

$$F_{\rho, A} = \min_{v \in \mathcal{F}_{\rho, A}} \int_{\Omega_\rho} | \nabla v |^2\ , \tag{8}$$

where

$$\mathcal{F}_{\rho, A} = \{v \in \mathcal{F}_\rho\, ; \int_{\partial G} | \frac{\partial v}{\partial \tau} |^2 \leq A\}\ .$$

We find an analogue of (5): if w_ρ is a solution of (8) then

$$\frac{1}{2} \int_{\Omega_\rho} | \nabla w_\rho |^2 = \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} + \widetilde{W}_A(a, \overline{d}) + o(1)\ , \quad \text{as } \rho \to 0\,,$$

where

$$\widetilde{W}_A(a,\overline{d}) = \inf\{W(a,\overline{d},g);\ \deg(g;\partial G) = d \ \text{ and } \int_{\partial G} \mid \frac{\partial g}{\partial \tau} \mid^2 \leq A\}$$

and the infimum is atteint.

In the last section we minimize the Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2}\int_G \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2}\int_G (1 - \mid u \mid^2)^2$$

in the class

$$\mathcal{H}_{d,A} = \{u \in H^1(G;\mathbf{R}^2);\ \mid u \mid = 1 \ \text{ on } \ \partial G,\ \deg(u,\partial G) = d \ \text{ and } \int_{\partial G} \mid \frac{\partial u}{\partial \tau} \mid^2 \leq A\}\ .$$

We prove that $\mathcal{H}_{d,A}$ is non-empty if A is sufficiently large and that the infimum of E_ε is achieved. If u_ε is a minimizer, we prove the convergence as $\varepsilon \to 0$ of u_ε to $u_\star$, which is a canonical harmonic map with values in S^1 and d singularities, say $a_1, \cdots, a_d$. Moreover, the configuration $a = (a_1, \cdots, a_d)$ minimizes the renormalized energy $\widetilde{W}_A$.

2. The renormalized energy for prescribed singularities and degrees

We recall that in [BBH4] the study of the minimization problems (1) and (4) is related to the unique solutions Φ_ρ , respectively $\hat{\Phi}_\rho$, of the following linear problems:

$$(9) \qquad \begin{cases} \Delta\Phi_\rho = 0 \quad \text{in } \Omega_\rho \\ \Phi_\rho = C_i = \text{Const.} \quad \text{on each } \ \partial\omega_i \text{ with } \ \omega_i = B(a_i,\rho) \\ \displaystyle\int_{\partial\omega_i} \frac{\partial\Phi_\rho}{\partial\nu} = 2\pi d_i \quad i = 1,\ldots k \\ \displaystyle\frac{\partial\Phi_\rho}{\partial\nu} = g \wedge g_\tau \quad \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_\rho = 0 \end{cases}$$

and

$$(10) \qquad \begin{cases} \Delta\hat{\Phi}_\rho = 0 \quad \text{in } \Omega \\ \hat{\Phi}_\rho = C_i = \text{Const.} \quad \text{on } \ \partial\omega_i \ i = 1,\ldots,k \\ \hat{\Phi}_\rho = 0 \quad \text{on } \partial G \\ \displaystyle\int_{\partial\omega_i} \frac{\partial\hat{\Phi}_\rho}{\partial\nu} = 2\pi d_i \quad i = 1,\ldots,k\ . \end{cases}$$

We also recall that Φ_ρ converges uniformly as $\rho \to 0$ to Φ_0, which is the unique solution of

$$(11) \qquad \begin{cases} \Delta\Phi_0 = 2\pi \sum_{j=1}^{k} d_j \delta_{a_j} \quad \text{in } G \\ \dfrac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau \quad \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_0 = 0 \ . \end{cases}$$

The explicit formula for $W(a, \overline{d}, g)$ found in [BBH4] is

$$(12) \qquad W(a, \overline{d}, g) = -\pi \sum_{i \neq j} d_i d_j \log \mid a_i - a_j \mid + \frac{1}{2} \int_{\partial G} \Phi_0 (g \wedge g_\tau) - \pi \sum_{i=1}^{k} d_i R_0(a_i) \ ,$$

where

$$R_0(x) = \Phi_0(x) - \sum_{j=1}^{k} d_j \log \mid x - a_j \mid \ .$$

We recall (see [BBH4]) that v is a canonical harmonic map with values in S^1 and boundary data g if it is harmonic and satisfies

$$\begin{cases} v \wedge \dfrac{\partial v}{\partial x_1} = -\dfrac{\partial \Phi_0}{\partial x_2} \quad \text{in } \ \Omega \\ v \wedge \dfrac{\partial v}{\partial x_2} = \dfrac{\partial \Phi_0}{\partial x_1} \quad \text{in } \ \Omega \ , \end{cases}$$

or, equivalently,

$$\frac{\partial}{\partial x_1}\left(v \wedge \frac{\partial v}{\partial x_1} \right) + \frac{\partial}{\partial x_2}\left(v \wedge \frac{\partial v}{\partial x_2} \right) = 0 \quad \text{in } \ \mathcal{D}'(G) \ .$$

If v is canonical and has singularities $a_1, \cdots, a_k \in G$ with topological degrees $d_1, \cdots, d_k$ then v has the form

$$v(z) = \left(\frac{z - a_1}{\mid z - a_1 \mid} \right)^{d_1} \cdots \left(\frac{z - a_k}{\mid z - a_k \mid} \right)^{d_k} e^{i\varphi(z)} \ ,$$

where φ is a uniquely determined smooth harmonic function in G.

We know from Chapter I in [BBH4] that

$$(13) \qquad \begin{cases} v_\rho \wedge \dfrac{\partial v_\rho}{\partial x_1} = -\dfrac{\partial \hat{\Phi}_\rho}{\partial x_2} \quad \text{in } \Omega_\rho \\ v_\rho \wedge \dfrac{\partial v_\rho}{\partial x_2} = \dfrac{\partial \hat{\Phi}_\rho}{\partial x_1} \quad \text{in } \Omega_\rho \ . \end{cases}$$

So

$$| \nabla v_\rho | = | \nabla \hat{\Phi}_\rho | \quad \text{in } \Omega_\rho \ . \tag{14}$$

Lemma 1. *$\hat{\Phi}_\rho$ converges to $\hat{\Phi}_0$ in $L^\infty(\Omega_\rho)$ as $\rho \to 0$. More precisely, there exists $C > 0$ such that*

$$\|\hat{\Phi}_\rho - \hat{\Phi}_0\|_{L^\infty(\Omega_\rho)} \le C\rho. \tag{15}$$

For the proof of Lemma 1 we need the following result of Bethuel, Brezis and Hélein (see [BBH4], Lemma I.4):

Lemma 2. *Let v be a solution of*

$$\begin{cases} \Delta v = 0 \quad \text{in } \Omega_\rho \\ v = 0 \quad \text{on } \partial G \\ \displaystyle\int_{\partial\omega_j} \frac{\partial v}{\partial \nu} = 0 \quad \text{for each } j \ . \end{cases} \tag{16}$$

Then

$$\sup_{\Omega_\rho} v - \inf_{\Omega_\rho} v \le \sum_{j=1}^{k} (\sup_{\omega_j} v - \inf_{\omega_j} v) \ .$$

Proof of Lemma 1. We apply Lemma 2 to the function $v = \hat{\Phi}_\rho - \hat{\Phi}_0$. Since $\hat{\Phi}_\rho = \text{Const.}$ on each $\partial B(a_j, \rho)$, it follows that

$$\sup_{\Omega_\rho} (\hat{\Phi}_\rho - \hat{\Phi}_0) - \inf_{\Omega_\rho} (\hat{\Phi}_\rho - \hat{\Phi}_0) \le \sum_{j=1}^{k} \left(\sup_{\partial B(a_j,\rho)} \hat{\Phi}_0 - \inf_{\partial B(a_j,\rho)} \hat{\Phi}_0 \right) \le C\rho \ .$$

Using now the fact that $\hat{\Phi}_\rho - \hat{\Phi}_0 = 0$ on ∂G we obtain

$$\|\hat{\Phi}_\rho - \hat{\Phi}_0\|_{L^\infty(\Omega_\rho)} \le C\rho \ . \tag{17}$$

□

Remark. By Lemma 1 and standard elliptic estimates it follows that $\hat{\Phi}_\rho$ converges in $C^k_{\text{loc}}(\Omega \cup \partial G)$ as $\rho \to 0$, for each $k \ge 0$.

Theorem 1. *As $\rho \to 0$ then (up to a subsequence) v_ρ converges in $C^k_{\mathrm{loc}}(\Omega \cup \partial G)$ to v_0, which is a canonical harmonic map.*

Moreover, the limits of two such sequences differ by a multiplicative constant of modulus 1.

Proof. We may write, locally on $\Omega_\rho \cup \partial G$, $v_\rho = e^{i\varphi_\rho}$ with $0 \le \varphi_\rho \le 2\pi$. Thus, by (13),

$$\text{(18)} \qquad \begin{cases} \dfrac{\partial \varphi_\rho}{\partial x_1} = -\dfrac{\partial \hat{\Phi}_\rho}{\partial x_2} & \text{in } \Omega_\rho \\ \dfrac{\partial \varphi_\rho}{\partial x_2} = \dfrac{\partial \hat{\Phi}_\rho}{\partial x_1} & \text{in } \Omega_\rho \ . \end{cases}$$

Hence, up to a subsequence, φ_ρ converges in $C^k_{\mathrm{loc}}(\Omega \cup \partial G)$. This means that v_ρ converges (up to a subsequence) in $C^k_{\mathrm{loc}}(\Omega \cup \partial G)$ to some v_0. Denote by $g_\rho = v_{\rho|\partial G}$. It is clear that g_ρ converges to some g_0 and v_0 satisfies

$$\text{(19)} \qquad \begin{cases} v_0 \wedge \dfrac{\partial v_0}{\partial x_1} = -\dfrac{\partial \hat{\Phi}_0}{\partial x_2} & \text{in } \Omega \\ v_0 \wedge \dfrac{\partial v_0}{\partial x_2} = \dfrac{\partial \hat{\Phi}_0}{\partial x_1} & \text{in } \Omega \\ v_0 = g_0 & \text{on } \partial G \ , \end{cases}$$

which means that v_0 is a canonical harmonic map.

We now consider two sequences $v_{\rho n}$ and $v_{\nu n}$ which converge to v_1 and v_2. Locally,

$$\varphi_{\rho_n} \to \varphi_1 \qquad \text{and} \qquad \varphi_{\nu_n} \to \varphi_2 \ .$$

Thus, $\nabla \varphi_1 = \nabla \varphi_2$, so φ_1 and φ_2 differ locally by an additive constant, which means that v_1 and v_2 differ locally by a multiplicative constant of modulus 1. By the connectedness of Ω, this constant is global. $\square$

Let

$$\hat{R}_0(x) = \hat{\Phi}_0(x) - \sum_{j=1}^{k} d_j \log \mid x - a_j \mid \ .$$

We observe that $\hat{R}_0$ is a smooth harmonic function in G.

Theorem 2. *We have the following asymptotic estimate:*

$$\text{(20)} \qquad \frac{1}{2}\int_{\Omega_\rho} \mid \nabla v_\rho \mid^2 = \pi \left(\sum_{j=1}^{k} d_j^2 \right) \log \frac{1}{\rho} + \widetilde{W}(a, \overline{d}) + O(\rho) \, , \quad \text{as } \rho \to 0 \ ,$$

where

$$\widetilde{W}(a,\overline{d}) = -\pi \sum_{i\neq j} d_i d_j \log | a_i - a_j | -\pi \sum_{j=1}^{k} d_j \hat{R}_0(a_j) \ . \tag{21}$$

Proof. We follow the ideas of the proof of Theorem I.7 in [BBH4].

Since $\hat{\Phi}_\rho$ is harmonic in Ω_ρ and $\hat{\Phi}_\rho = 0$ on ∂G we may write

$$\frac{1}{2}\int_{\Omega_\rho} | \nabla v_\rho |^2 = \frac{1}{2}\int_{\Omega_\rho} | \nabla \hat{\Phi}_\rho |^2 = -\frac{1}{2}\sum_{j=1}^{k}\int_{\partial B(a_j,\rho)} \frac{\partial \hat{\Phi}_\rho}{\partial \nu}\, \hat{\Phi}_\rho = -\pi \sum_{j=1}^{k} d_j\, \hat{\Phi}_\rho\Big(\partial B(a_j,\rho)\Big) \ .$$

By Lemma 1 and the expression of $\hat{R}_0$ we easily deduce (20). □

Theorem 3. *The following equality holds:*

$$\widetilde{W}(a,\overline{d}) = \inf_{\deg\,(g;\partial G)=d} W(a,\overline{d},g) \tag{22}$$

and the infimum is achieved.

Proof. *Step 1.* $\widetilde{W}(a,\overline{d}) \leq \inf_{\deg\,(g;\partial G)=d} W(a,\overline{d},g)$.

Suppose not, then there exist $\varepsilon > 0$ and $g : \partial G \to S^1$ with $\deg\,(g;\partial G) = d$ such that

$$W(a,\overline{d},g) + \varepsilon \leq \widetilde{W}(a,\overline{d}) \ . \tag{23}$$

Thus, if u_ρ is a solution of (1), then

$$\frac{1}{2}\int_{\Omega_\rho} | \nabla u_\rho |^2 = \pi \left(\sum_{j=1}^{k} d_j^2\right) \log\frac{1}{\rho} + W(a,\overline{d},g) + O(\rho) \geq \tag{24}$$

$$\geq \frac{1}{2}\int_{\Omega_\rho} | \nabla v_\rho |^2 = \pi \left(\sum_{j=1}^{k} d_j^2\right) \log\frac{1}{\rho} + \widetilde{W}(a,\overline{d}) + O(\rho) \ , \quad \text{as } \rho \to 0 \ .$$

We obtain a contradiction by (23) and (24).

Step 2. If g_ρ and g_0 are as in the proof of Theorem 1, then

$$\widetilde{W}(a,\overline{d}) = W(a,\overline{d},g_0) \ .$$

For $r > 0$ let $u_{\rho,r}$ be a solution of the minimization problem

$$\min_{u\in\mathcal{E}_{r,g_\rho}} \int_{\Omega_r} | \nabla u |^2 \ . \tag{25}$$

Denote $u_{\rho,\rho} = u_\rho$ and $\Phi_{\rho,r}$ the solution of the associated linear problem (see (9)). Let $\Phi_{\rho,0}$ be the solution of (11) for g replaced by g_ρ.

We recall (see Theorem I.6 in [BBH4]) that

$$\Phi_{\rho,r} \to \Phi_{\rho,0} \qquad \text{in } C^k_{\text{loc}}(\Omega \cup \partial G) \ \text{ as } r \to 0 \tag{26}$$

and

$$\left| \ \frac{1}{2}\int_{\Omega_r} |\nabla u_{\rho,r}|^2 - \pi\left(\sum_{j=1}^k d_j^2\right) \log\frac{1}{r} - W(a,\overline{d},g_\rho) \right| \leq C_{g_\rho} r\,, \tag{27}$$

where $C_g = C(g) > 0$ is a constant which depends on the boundary data g.

Our aim is to prove that C_{g_ρ} is uniformly bounded for $\rho > 0$. Indeed, analysing the proof of Theorem I.7 in [BBH4] we observe that C_{g_ρ} depends on $\widetilde{C}_{g_\rho}$, which appears in

$$\|\Phi_{\rho,r} - \Phi_{\rho,0}\|_{L^\infty(\Omega_r)} \leq \sum_{j=1}^k \left[\sup_{\partial B(a_j,r)} \Phi_{\rho,0} - \inf_{\partial B(a_j,r)} \Phi_{\rho,0}\right] \leq \widetilde{C}_{g_\rho} r\ . \tag{28}$$

It is clear at this stage, by the convergence of g_ρ and elliptic estimates, that $\widetilde{C}_{g_\rho}$ is uniformly bounded.

Observe now that the map $C^1(\partial G; S^1) \ni g \longmapsto W(a,\overline{d},g)$ is continuous. We have

$$| W(a,\overline{d},g_0) - \widetilde{W}(a,\overline{d}) | \leq \left| \frac{1}{2}\int_{\Omega_\rho} |\nabla v_\rho|^2 - \pi\left(\sum_{j=1}^k d_j^2\right) \log\frac{1}{\rho} - \widetilde{W}(a,\overline{d}) \right| +$$

$$+ \left| \frac{1}{2}\int_{\Omega_\rho} |\nabla v_\rho|^2 - \pi\left(\sum_{j=1}^k d_j^2\right) \log\frac{1}{\rho} - W(a,\overline{d},g_\rho) \right| + | W(a,\overline{d},g_\rho) - W(a,\overline{d},g_0) | \leq$$

$$\leq O(\rho) + C\rho + | W(a,\overline{d},g_\rho) - W(a,\overline{d},g_0) | \to 0 \qquad \text{as } \rho \to 0\ .$$

Thus

$$\widetilde{W}(a,\overline{d}) = W(a,\overline{d},g_0)\ ,$$

which concludes the proof of Step 2. □

Theorem 4. *For fixed A, if w_ρ is a solution of the minimization problem (8) then the following holds:*

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla w_\rho|^2 = \pi\left(\sum_{j=1}^k d_j^2\right) \log\frac{1}{\rho} + \widetilde{W}_A(a,\overline{d}) + o(1)\ , \quad \text{as } \rho \to 0\,, \tag{29}$$

where

$$(30) \qquad \widetilde{W}_A(a,\overline{d}) = \inf\{W(a,\overline{d},g);\ \deg(g;\partial G)=d \ \text{ and } \ \int_{\partial G} \mid \frac{\partial g}{\partial\tau} \mid^2 \le A\}\ ,$$

and the infimum is achieved.

Moreover, w_ρ converges in $C^{0,\alpha}_{\text{loc}}(\Omega\cup\partial G)$ to the canonical harmonic map associated to g_0, a, $\overline{d}$.

Proof. The existence of w_ρ is obvious. Let $g_\rho = w_\rho \mid_{\partial G}$. It follows from Chapter I in [BBH4] that

$$(31) \qquad \mid \frac{1}{2}\int_{\Omega_\rho} \mid \nabla w_\rho \mid^2 - \pi\left(\sum_{j=1}^k d_j^2\right) \log\frac{1}{\rho} + W(a,\overline{d},g_\rho) \mid \le C_{g_\rho}\cdot\rho\ , \quad \text{as} \ \ \rho\to 0\ ,$$

where C_g depends only on g, a and $\overline{d}$.

By the boundedness of g_ρ in $H^1(\partial G)$ we may suppose that (up to a subsequence)

$$g_\rho \rightharpoonup g_0 \qquad \text{weakly in} \ \ H^1(\partial G), \ \ \text{as} \ \ \rho\to 0\ .$$

As in the proof of Theorem 3 (see (28)) we deduce that C_{g_ρ} is uniformly bounded.

We now prove that the map $g \longmapsto W(a,\overline{d},g)$ is continuous in the weak topology of $H^1(\partial G)$. Taking into account the weak convergence of g_ρ to g_0 and the Sobolev embedding Theorem we obtain

$$g_\rho\wedge\frac{\partial g_\rho}{\partial\tau} \rightharpoonup g_0\wedge\frac{\partial g_0}{\partial\tau} \qquad \text{weakly in} \ \ L^2(\partial G), \ \ \text{as} \ \ \rho\to 0\ .$$

Using (11), it follows that

$$\Phi_{\rho,0} \rightharpoonup \Phi_0 \qquad \text{weakly in} \ \ H^1(G), \ \text{as}\ \rho\to 0\ .$$

So, by the Rellich Theorem,

$$\Phi_{\rho,0} \to \Phi_0 \qquad \text{strongly in} \ \ L^2(G), \ \ \text{as} \ \ \rho\to 0\ .$$

Therefore,

$$\int_{\partial G} \Phi_{\rho,0}\left(g_\rho\wedge\frac{\partial g_\rho}{\partial\tau}\right) \to \int_{\partial G}\Phi_0\left(g_0\wedge\frac{\partial g_0}{\partial\tau}\right) \quad \text{as}\ \rho\to 0\ .$$

We also deduce, using elliptic estimates, that for each i,

$$R_{\rho,0}(a_i)\to R_0(a_i) \qquad \text{as} \ \ \rho\to 0\ .$$

Thus, by (12), we obtain the continuity of the map $g\longmapsto W(a,\overline{d},g)$. Hence, by (31), we easily deduce (29).

The fact that the infimum in (30) is achieved may be deduced with similar arguments as in the proof of Theorem 3.

The convergence of w_ρ to a canonical harmonic map follows easily from the convergence of g_ρ. □

3. Renormalized energies in a particular case

We shall calculate in the first part of this section the expressions of $\widetilde{W}(a,\overline{d})$ and $\widetilde{W}(a,\overline{d},g)$ when $G=B(0;1)$ and $g(\theta)=e^{id\theta}$, for an arbitrary configuration $a=(a_1,...,a_k)$.

Proposition 1. *The expression of the renormalized energy $\widetilde{W}(a,\overline{d})$ is given by*

$$\widetilde{W}(a,\overline{d}) = -\pi\sum_{i\neq j} d_i d_j \,\log\mid a_i-a_j\mid + \pi\sum_{i,j} d_i d_j\,\log\mid 1-a_i\overline{a}_j\mid \ .$$

Proof. Let $\hat{R}_0$ be defined as in the preceding section. Then

$$\begin{cases} \Delta\hat{R}_0 = 0 & \text{in } B_1 \\ \hat{R}_0(x) = -\displaystyle\sum_{j=1}^{k} d_j\,\log\mid x-a_j\mid & \text{if } x\in\partial B_1\ . \end{cases}$$

It follows from the linearity of this problem that it is sufficient to compute $\hat{R}_0$ when the configuration of points consists of one point, say a. Hence, by the Poisson formula, for each $x\in B_1$,

$$\hat{R}_0(x) = -\frac{d}{2\pi}\,(1-\mid x\mid^2)\int_{\partial B_1}\frac{\log\mid z-a\mid}{\mid z-x\mid^2}dz\ . \tag{32}$$

We first observe that

$$\hat{R}_0(x)=0 \qquad \text{if } \ a=0\ . \tag{33}$$

If $a\neq 0$ and $a^\star = \dfrac{a}{\mid a\mid^2}$, then

$$\begin{aligned} \hat{R}_0(x) &= -\frac{d}{2\pi}\,(1-\mid x\mid^2)\int_{\partial B_1}\frac{\log\mid z-a^\star\mid+\log\mid a\mid}{\mid z-x\mid^2}dz = \\ &= -d\log\mid x-a^\star\mid -d\log\mid a\mid\ . \end{aligned} \tag{34}$$

Hence, by (33) and (34)

$$\hat{R}_0(x) = \begin{cases} 0 & \text{if } a=0 \\ -\,d\log\mid x-a^\star\mid -d\log\mid a\mid & \text{if } a\neq 0\ . \end{cases} \tag{35}$$

In the case of a general configuration $a=(a_1,...,a_k)$ one has

$$\hat{R}_0(x) = -\sum_{j=1}^{k} d_j\log\mid x-a_j^\star\mid - \sum_{j=1}^{k} d_j\log\mid a_j\mid\ . \tag{36}$$

Applying now Theorem 2 we obtain

$$\widetilde{W}(a,\overline{d}) = -\pi \sum_{i \neq j} d_i d_j \log \mid a_i - a_j \mid + \pi \sum_{i,j} d_i d_j \log \mid 1 - a_i \overline{a}_j \mid \ .$$

□

Proposition 2. *The expression of $W(a,\overline{d},g)$ if $G = B_1$ and $g(\theta) = e^{id\theta}$ is given by*

$$W(a,\overline{d},g) = -\pi \sum_{i \neq j} d_i d_j \log \mid a_i - a_j \mid - \pi \sum_{i,j} d_i d_j \log \mid 1 - a_i \overline{a}_j \mid \ . \tag{37}$$

Proof. We shall use the expression (12) for the renormalized energy $W(a,\overline{d},g)$. As above, we observe that it suffices to compute R_0 for one point, say a.

We define on $B(0;1) \setminus \{a\}$ the function $\mathcal{G}$ by

$$\mathcal{G}(x) = \begin{cases} \dfrac{d}{2\pi} \log \mid x - a \mid + \dfrac{d}{2\pi} \log \mid x - a^\star \mid - \dfrac{d}{4\pi} \mid x \mid^2 + \mathcal{C} & \text{if } a \neq 0 \\ \dfrac{d}{2\pi} \log \mid x \mid - \dfrac{d}{4\pi} \mid x \mid^2 + \mathcal{C} & \text{if } a = 0 \end{cases} \tag{38}$$

and we choose the constant $\mathcal{C}$ such that

$$\int_{\partial B_1} \mathcal{G} = 0 \ .$$

It follows that, for every $a \in B_1$,

$$\mathcal{C} = \frac{d}{4\pi} + \frac{d}{2\pi} \log \mid a \mid \ . \tag{39}$$

The function $\mathcal{G}$ satisfies

$$\begin{cases} \Delta \mathcal{G} = d\delta_a - \dfrac{d}{\pi} & \text{in } B_1 \\ \dfrac{\partial \mathcal{G}}{\partial \nu} = 0 & \text{on } \partial B_1 \\ \displaystyle\int_{\partial B_1} \mathcal{G} = 0 \ . \end{cases} \tag{40}$$

It follows now from (11) that

$$\begin{cases} \Delta \left(\dfrac{\Phi_0}{2\pi} \right) = d\delta_a & \text{in } B_1 \\ \dfrac{\partial}{\partial \nu} \left(\dfrac{\Phi_0}{2\pi} \right) = \dfrac{d}{2\pi} & \text{on } \partial B_1 \\ \displaystyle\int_{\partial B_1} \dfrac{\Phi_0}{2\pi} = 0 \ . \end{cases}$$

Thus the function $\Psi = \frac{\Phi_0}{2\pi} - \frac{d}{4\pi}(\mid x \mid^2 -1)$ satisfies

$$
\begin{cases} \Delta\Psi = d\delta_a - \frac{d}{\pi} \quad \text{in } B_1 \\ \frac{\partial\Psi}{\partial\nu} = 0 \quad \text{on } \partial B_1 \\ \int_{\partial B_1} \Psi = 0 \ . \end{cases} \tag{41}
$$

By uniqueness, it follows from (40) and (41) that

$$
\frac{\Phi_0}{2\pi} - \frac{d}{4\pi}(\mid x \mid^2 -1) = \frac{d}{2\pi}\log \mid x-a \mid + \frac{d}{2\pi}\log \mid x - a^\star \mid - \frac{d}{4\pi} \mid x \mid^2 + \mathcal{C} \ . \tag{42}
$$

Taking into account the expression of $\mathcal{C}$ given in (39), as well as the link between Φ_0 and R_0 we obtain (37). □

Remark. It follows by Theorem 3 and Propositions 1 and 2 that

$$
\sum_{i\neq j} d_i d_j \log \mid a_i - a_j \mid + \sum_{j=1}^{k} d_j^2 \log(1- \mid a_j \mid^2) \leq 0 \ .
$$

A very interesting problem is the study of configurations which minimize $W(a, \overline{d}, g)$ with $\overline{d}$ and g prescribed. This relies on the behavior of minimizers of the Ginzburg-Landau energy (see [BBH4] for further details).

Proposition 3. *If $k = 2$ and $d_1 = d_2 = 1$, then the minimal configuration for W is unique (up to a rotation) and consists of two points which are symmetric with respect to the origin.*

Proof. Let a and b be two distinct points in B_1. Then

$$
-\frac{W}{\pi} = \log(\mid a \mid^2 + \mid b \mid^2 -2 \mid a \mid \cdot \mid b \mid \cdot \cos\varphi) + \log(1+ \mid a \mid^2 \mid b \mid^2 -2 \mid a \mid \cdot \mid b \mid \cdot \cos\varphi) +
$$

$$
+ \log(1- \mid a \mid^2) + \log(1- \mid b \mid^2) \ ,
$$

where φ denotes the angle between the vectors $\overrightarrow{Oa}$ and $\overrightarrow{Ob}$. So, a necessary condition for the minimum of W is $\cos\varphi = -1$, that is the points a, O and b are colinear, with O between a and b. Hence one may suppose that the points a and b lie on the real axis and $-1 < b < 0 < a < 1$. Denote

$$
f(a,b) = 2\log(a-b) + 2\log(1-ab) + \log(1-a^2) + \log(1-b^2) \ .
$$

A straightforward calculation, based on the Jensen inequality and the symmetry of f, shows that $a = -b = 5^{-1/4}$. $\square$

4. The behavior of minimizers of the Ginzburg-Landau energy

We assume throughout this section that G is strictly starshaped about the origin.

In [BBH2] and [BBH4], F. Bethuel, H. Brezis and F. Hélein studied the behavior of minimizers of the Ginzburg-Landau energy E_ε in

$$H^1_g(G;\mathbf{R}^2) = \{u \in H^1(G;\mathbf{R}^2);\ u = g \ \text{ on } \ \partial G\}\ ,$$

for some smooth fixed $g : \partial G \to S^1$, $\deg(g;\partial G) = d > 0$. Our aim is to study a similar problem, that is the behavior of minimizers u_ε of E_ε in the class

$$\mathcal{H}_{d,A} = \{u \in H^1(G;\mathbf{R}^2);\ \mid u \mid = 1 \text{ on } \partial G,\ \deg(u,\partial G) = d \text{ and } \int_{\partial G} \mid \frac{\partial u}{\partial \tau} \mid^2 \leq A\}\ . \tag{43}$$

It would have seemed more natural to minimize E_ε in the class

$$\mathcal{H}_d = \{u \in H^1(G;\mathbf{R}^2);\ \mid u \mid = 1 \text{ on } \partial G,\ \deg(u,\partial G) = d\}$$

but, as observed by F. Bethuel, H. Brezis and F. Hélein, the infimum of E_ε is not atteint. To show this, they consider the particular case when $G = B_1$, $d = 1$ and $g(x) = x$. This is the reason why we take the infimum of E_ε on the class $\mathcal{H}_{d,A}$, that was also considered by F. Bethuel, H. Brezis and F. Hélein.

Theorem 5. *For each sequence $\varepsilon_n \to 0$, there is a subsequence (also denoted by ε_n) and exactly d points $a_1, \cdots, a_d$ in G such that*

$$u_{\varepsilon_n} \to u_\star \quad \text{in } H^1_{\text{loc}}(\overline{G} \setminus \{a_1, \cdots, a_d\};\mathbf{R}^2)\ ,$$

where $u_\star$ is a canonical harmonic map with values in S^1 and singularities $a_1, \cdots, a_d$ of degrees +1.

Moreover, the configuration $a = (a_1, \cdots, a_d)$ is a minimum point of

$$\widetilde{W}_A(a,\overline{d}) := \min\{W(a,\overline{d},g);\ \deg(g;\partial G) = d \ \text{ and } \int_{\partial G} \mid \frac{\partial g}{\partial \tau} \mid^2 \leq A\}\ .$$

Proof. *Step 1.* The existence of u_ε.

For fixed ε, let u_ε^n be a minimizing sequence for E_ε in $\mathcal{H}_{d,A}$. It follows that (up to a subsequence)

$$u_\varepsilon^n \rightharpoonup u_\varepsilon \quad \text{weakly in } H^1$$

and, by the boundedness of $u_\varepsilon^n \mid_{\partial G}$ in $H^1(\partial G)$, we obtain that

$$u_{\varepsilon_n} \mid_{\partial G} \to u_\varepsilon \mid_{\partial G} \quad \text{strongly in } H^{\frac{1}{2}}(\partial G) \ .$$

This means that, if $g_\varepsilon = u_\varepsilon \mid_{\partial G}$, then

$$\deg\left(g_\varepsilon; \partial G\right) = d \ .$$

By the lower semi-continuity of E_ε, u_ε is a minimizer of E_ε. Moreover, this u_ε satisfies the Ginzburg-Landau equation

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - \mid u_\varepsilon \mid^2) \quad \text{in } G \ . \tag{44}$$

Step 2. A fundamental estimate.

As in the proof of Theorem III.2 in [BBH4], multiplying (47) by $x \cdot \nabla u_\varepsilon$ and integrating on G, we find

$$\frac{1}{2}\int_{\partial G} (x\cdot\nu)\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)^2 + \frac{1}{2\varepsilon^2}\int_G (1 - \mid u_\varepsilon \mid^2)^2 = \tag{45}$$

$$= \frac{1}{2}\int_{\partial G} (x\cdot\nu)\left(\frac{\partial g_\varepsilon}{\partial \tau}\right)^2 - \int_{\partial G} (x\cdot\tau)\frac{\partial u_\varepsilon}{\partial \nu}\frac{\partial g_\varepsilon}{\partial \tau} \ .$$

Using now the boundedness of g_ε in $H^1(\partial G)$ and the fact that G is strictly starshaped we easily obtain

$$\int_{\partial G} \mid \frac{\partial u_\varepsilon}{\partial \nu} \mid^2 + \frac{1}{\varepsilon^2}\int_G (1 - \mid u_\varepsilon \mid^2)^2 \le C \ , \tag{46}$$

where C depends only on A and d.

Step 3. A fundamental Lemma.

The following result is an adapted version of Theorem III.3 in [BBH4] which is essential towards locating the singularities at the limit.

Lemma 3. *There exist positive constants λ_0 and μ_0 (which depend only on G, d and A) such that if*

$$\frac{1}{\varepsilon^2}\int_{G\cap B_{2\ell}} (1 - \mid u_\varepsilon \mid^2)^2 \le \mu_0 \ ,$$

where $B_{2\ell}$ is some disc of radius 2ℓ in $\mathbf{R}^2$ with

$$\frac{\ell}{\varepsilon} \geq \lambda_0 \quad \textit{and} \quad \ell \leq 1 \ ,$$

then

$$| \ u_\varepsilon(x) \ | \geq \frac{1}{2} \quad \textit{if } x \in G \cap B_\ell \ . \tag{47}$$

The proof of Lemma is essentially the same as of the cited theorem, after observing that

$$\|\nabla u_\varepsilon\|_{L^\infty(G)} \leq \frac{C}{\varepsilon} \ ,$$

where C depends only on G, d and A.

Step 4. The convergence of u_ε.

Using Lemma 1 and the estimate (46), we may apply the methods developed in Chapters III-V in [BBH4] to determine the "bad" discs, as well as the fact that their number is uniformly bounded. The same techniques allow us to prove the weak convergence in $H^1_{\mathrm{loc}}(G \setminus \{a_1, \cdots, a_k\}; \mathbf{R}^2)$ of a subsequence, also denoted by u_{ε_n}, to some $u_\star$.

As in [BBH4], Chapter X (see also [S]) one may prove that, for each $p < 2$,

$$u_{\varepsilon_n} \to u_\star \quad \text{in } W^{1,p}(G) \ .$$

This allows us to pass at the limit in

$$\frac{\partial}{\partial x_1}\left(u_{\varepsilon_n} \wedge \frac{\partial u_{\varepsilon_n}}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(u_{\varepsilon_n} \wedge \frac{\partial u_{\varepsilon_n}}{\partial x_2}\right) = 0 \quad \text{in } \ \mathcal{D}'(G)$$

and to deduce that $u_\star$ is a canonical harmonic map.

The strong convergence of (u_{ε_n}) in $H^1_{\mathrm{loc}}(\overline{G} \setminus \{a_1, \cdots, a_k\}; \mathbf{R}^2)$ follows as in [BBH4], Theorem VI.1 with the techniques from [BBH3], Theorem 2, Step 1.

We then observe that for all j, $\deg(u_\star, a_j) \neq 0$. Indeed, if not, then as in Step 1 of Theorem 2 in [BBH3], the H^1-convergence is extended up to a_j, which becomes a "removable singularity". The fact that all these degrees equal $+1$ and the points $a_1, \cdots, a_d$ are not on the boundary may be deduced as in Theorem VI.2 [BBH4].

The following steps are devoted to characterize the limiting configuration as a minimum point of the renormalized energy $\widetilde{W}_A$.

Step 5. An upper bound for $E_\varepsilon(u_\varepsilon)$.

For $R > 0$, let $I(R)$ be the infimum of E_ε on $H^1_g(G)$ with $G = B(0; \frac{\varepsilon}{R})$ and $g(x) = \frac{x}{\mid x \mid}$ on ∂G. Following the ideas of the proof of Lemma VIII.1 in [BBH4] one may show that if $b = (b_j)$ is an arbitrary configuration of d distinct points in G and g is such that $\deg(g, \partial G) = d$ and $\int_{\partial G} \mid \frac{\partial g}{\partial \tau} \mid^2 \leq A$, then there exists $\eta_0 > 0$ such that, for each $\eta < \eta_0$,

$$E_\varepsilon(u_\varepsilon) \leq dI\left(\frac{\varepsilon}{\eta}\right) + W(b, g) + \pi d \log \frac{1}{\eta} + O(\eta) \ , \quad \text{as } \eta \to 0 \tag{48}$$

for $\varepsilon > 0$ small enough. Here $O(\eta)$ stands for a quantity which is bounded by C_η, where C is a constant depending only on g.

Step 6. A lower bound for $E_{\varepsilon_n}(u_{\varepsilon_n})$.

With the same proof as of Step 2 of Theorem 1 in [LR] one may show that if $a_1, \cdots, a_d$ are the singularities of $u_\star$ and $\eta > 0$, then there is $N_0 = N_0(\eta) \in \mathbf{N}$ such that, for each $n \geq N_0$,

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq dI\left(\frac{\varepsilon_n}{\eta(1+\eta)}\right) + \pi d \log \frac{1}{\eta} + W(a, g_0) + O(\eta) \ , \tag{49}$$

where $O(\eta)$ is a quantity bounded by $C\eta$, where C depends only on g_0.

Step 7. The limiting configuration is a minimum point for $\widetilde{W}_A$.

Taking into account that (see [BBH4], Chapter III)

$$I(\varepsilon) = \pi \ \mid \log \varepsilon \mid + \gamma + O(\varepsilon) \ ,$$

we obtain by (48) and (49)

$$\begin{aligned} & W(b, g) - \pi d \log \varepsilon_n + d\gamma + O\left(\frac{\varepsilon_n}{\eta}\right) \geq \\ & \geq W(a, g_0) - \pi d \log \varepsilon_n + d\gamma + O(\eta) \ . \end{aligned} \tag{50}$$

Adding $\pi d \log \varepsilon_n$ in (50) and passing to the limit firstly as $n \to \infty$ and then as $\eta \to 0$, we find

$$W(a, g_0) \leq W(b, g) \ . \tag{51}$$

As b and g are arbitrary chosen it follows that $a = (a_1, \cdots, a_d)$ is a global minimum point of

$$\widetilde{W}_A(b) = \min \left\{ W(b, g); \ \deg(g; \partial G) = d \ \text{ and } \ \int_{\partial G} \mid \frac{\partial g}{\partial \tau} \mid^2 \leq A \right\} \ . \tag{52}$$

$\square$

Remark. The infimum in (52) is achieved because of the continuity of the mapping $\mathcal{H}_{d,A} \ni g \longmapsto W(b,g)$ with respect to the weak topology of $H^1(\partial G)$.

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THE RENORMALIZED ENERGY ASSOCIATED TO A HARMONIC MAP

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Introduction

In [BBH2], F. Bethuel, H. Brezis and F. Hélein have studied several problems which occur in superconductivity and superfluids and they have introduced the notion of renormalized energy. We recall the essential facts: Let $G \subset \mathbf{R}^2$ be a smooth simply connected bounded domain and let $g : \partial G \to S^1$ be a smooth map of topological degree $d > 0$. Consider a configuration $a = (a_1, ..., a_k)$ of distinct points in G and $\overline{d} = (d_1, ..., d_k) \in \mathbf{Z}^k$ such that $d_1 + ... + d_k = d$. The *canonical harmonic map* $u_0 : \Omega = G \setminus \{a_1, ...a_k\} \to S^1$ associated to $(a, \overline{d}, g)$ is defined by

$$u_0(z) = \left(\frac{z - a_1}{\mid z - a_1 \mid}\right)^{d_1} \cdot ... \cdot \left(\frac{z - a_k}{\mid z - a_k \mid}\right)^{d_k} \cdot e^{i\varphi_0(z)} \quad \text{if } \ z \in G, \tag{1}$$

where

$$\begin{cases} \Delta\varphi_0 = 0 & \text{in } \ G \\ u_0 = g & \text{on } \ \partial G \ . \end{cases}$$

For each $\rho > 0$ sufficiently small we define

$$G_\rho = G \setminus \bigcup_{j=1}^{k} \overline{B(a_j, \rho)} \ .$$

The renormalized energy $W(a, \overline{d}, g)$ appears in Chapter I of [BBH2] as

$$W(a, \overline{d}, g) = \lim_{\rho \to 0} \left\{ \frac{1}{2} \int_{G_\rho} \mid \nabla u_0 \mid^2 - \pi \left(\sum_{j=1}^{k} d_j^2 \right) \log \frac{1}{\rho} \right\} \ . \tag{2}$$

We also recall that any *harmonic map* $u : \Omega \to S^1, u = g$ on ∂G with $\deg(u, a_j) = d_j$ has the form

$$u = e^{i\psi} u_0 \quad \text{ on } \Omega \tag{3}$$

where

$$\begin{cases} \psi(x) = \displaystyle\sum_{j=1}^{k} c_j \log \mid x - a_j \mid + \phi(x) \\ \psi = 0 \quad \text{on } \partial G \\ \Delta\phi = 0 \quad \text{ on } G. \end{cases} \tag{4}$$

In the first section we define a notion of renormalized energy associated to a harmonic map u, which coincides with $W(a, \overline{d}, g)$ when $u = u_0$. In the second part of this paper we give an explicit formula for our notion of renormalized energy.

1. The main result

Theorem 1. *For any harmonic map $u : \Omega \to S^1$ of the form (3) the following limit exists and is finite*

$$\lim_{p \nearrow 2} \left\{ \frac{1}{2} \int_G \mid \nabla u \mid^p - \frac{\pi}{2 - p} \sum_{j=1}^{k} (c_j^2 + d_j^2) \right\} + \frac{\pi}{2} \sum_{j=1}^{k} (c_j^2 + d_j^2) \cdot \log \left(\sum_{j=1}^{k} (c_j^2 + d_j^2) \right) =: W(u) \tag{5}$$

Moreover

$$W(u) = \lim_{\rho \to 0} \left\{ \frac{1}{2} \int_{G_\rho} \mid \nabla u \mid^2 - \pi \left(\sum_{j=1}^{k} (c_j^2 + d_j^2) \right) \log \frac{1}{\rho} \right\}. \tag{6}$$

Proof. Fix $\rho > 0$ such that the closed balls $\overline{B(a_j, \rho)}$ are mutually disjoint and included in G.

We shall estimate ∇u in the neighbourhood of a singularity a_j, supposed to be 0. There exists a smooth harmonic function ζ such that, if $0 <\mid x \mid \leq \rho$,

$$u(x) = e^{i(c_j \log|x| + d_j \theta + \zeta(x))} .$$

Hence

$$\mid \nabla u \mid = \mid \nabla(c_j \log \mid x \mid + d_j\theta + \zeta) \mid = \tag{7}$$

$$=\left| \frac{c_j}{|x|}(\cos\theta,\sin\theta)+\frac{d_j}{|x|}(-\sin\theta,\cos\theta)+\nabla\zeta \right| = \left| \frac{\sqrt{c_j^2+d_j^2}}{|x|}e^{i(\theta+\theta_0)}+\nabla\zeta \right| ,$$

where $\theta_0 \in [0,2\pi)$ depends only on c_j and d_j.

We observe that the term $\nabla\zeta$ is negligible in $\int_{B(0,\rho)} |\nabla u|^p$, in the sense that

$$\int_{B(0,\rho)} \left| |\nabla u|^p - \left| \frac{\sqrt{c_j^2+d_j^2}}{|x|}e^{i(\theta+\theta_0)} \right|^p \right| \leq \text{(The Mean Value Theorem)} \tag{8}$$

$$\leq C\int_{B(0,\rho)} \frac{1}{r^{p-1}}dx = O(\rho) \quad \text{as } p \nearrow 2 .$$

Therefore

$$\frac{1}{2}\int_G |\nabla u|^p - \frac{\pi}{2-p}\sum_{j=1}^k (c_j^2+d_j^2)^{\frac{p}{2}} =$$

$$= \frac{1}{2}\int_{G_\rho} |\nabla u|^p + \sum_{j=1}^k \left[\frac{1}{2}\int_{B(a_j,\rho)} |\nabla u|^p - \frac{\pi}{2-p}(c_j^2+d_j^2)^{\frac{p}{2}}\right] \leq$$

$$\leq \frac{1}{2}\int_{G_\rho} |\nabla u|^p + \sum_{j=1}^k \left[\frac{\pi}{2-p}(c_j^2+d_j^2)^{\frac{p}{2}}\rho^{2-p} - \frac{\pi}{2-p}(c_j^2+d_j^2)^{\frac{p}{2}}\right] + C_1\rho , \quad \text{as } p \nearrow 2 ,$$

for some fixed constant C_1.

It follows that

$$\limsup_{p\nearrow 2}\left\{\frac{1}{2}\int_G |\nabla u|^p - \frac{\pi}{2-p}\sum_{j=1}^k (c_j^2+d_j^2)^{\frac{p}{2}}\right\} \leq \tag{9}$$

$$\leq \frac{1}{2}\int_{G_\rho} |\nabla u|^2 - \pi\left(\sum_{j=1}^k (c_j^2+d_j^2)\right)\log\frac{1}{\rho} + C_1\rho .$$

At the same manner we can find a constant C_2 such that

$$\liminf_{p\nearrow 2}\left\{\frac{1}{2}\int_G |\nabla u|^p - \frac{\pi}{2-p}\sum_{j=1}^k (c_j^2+d_j^2)^{\frac{p}{2}}\right\} \geq \tag{10}$$

$$\geq \frac{1}{2}\int_{G_\rho} |\nabla u|^2 - \pi\left(\sum_{j=1}^k (c_j^2+d_j^2)\right)\log\frac{1}{\rho} - C_2\rho .$$

The relations (9) and (10) show that the two limits are finite and their difference is $O(\rho)$. Since ρ is arbitrary, it follows that the limit in (5) exists and is finite.

Now we can also deduce from (9) and (10) that (6) holds. $\square$

Corollary 1. *For each u as in Theorem 1,*

$$\lim_{p \nearrow 2} (2-p) \int_G \mid \nabla u \mid^p = 2\pi \sum_{j=1}^{k} (c_j^2 + d_j^2) \ .$$

The proof of this equality follows obviously from Theorem 1.

Corollary 2. *If u_0 is the canonical harmonic map associated to $(a, \overline{d}, g)$ then*

$$W(u_0) = W(a, \overline{d}, g) \ .$$

The proof follows immediately from (6).

2. An explicit formula for the renormalized energy

Our purpose in what follows is to give an explicit formula for the renormalized energy $W(u)$, for any harmonic map $u : \Omega \to S^1$. To do this, we shall use the asymptotic evaluate given by (6).

It follows by (3) that

$$\text{(11)} \qquad \begin{cases} u \wedge \dfrac{\partial u}{\partial x_1} = u_0 \wedge \dfrac{\partial u_0}{\partial x_1} + \dfrac{\partial \psi}{\partial x_1} & \text{in } \Omega \\ u \wedge \dfrac{\partial u}{\partial x_2} = u_0 \wedge \dfrac{\partial u_0}{\partial x_2} + \dfrac{\partial \psi}{\partial x_2} & \text{in } \Omega \ . \end{cases}$$

We recall (see Chapter 1 in [BBH2]) that

$$\text{(12)} \qquad \begin{cases} u_0 \wedge \dfrac{\partial u_0}{\partial x_1} = -\dfrac{\partial \Phi_0}{\partial x_2} & \text{in } \Omega \\ u_0 \wedge \dfrac{\partial u_0}{\partial x_2} = \dfrac{\partial \Phi_0}{\partial x_1} & \text{in } \Omega \ , \end{cases}$$

where Φ_0 is the (unique) solution of

$$\begin{cases} \Delta \Phi_0 = 2\pi \displaystyle\sum_{j=1}^{k} d_j \delta_{a_j} & \text{in } G \\ \dfrac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau & \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_0 = 0 \ . \end{cases}$$

Inserting (12) into (11) we obtain

$$(13)\qquad \begin{cases} u \wedge \dfrac{\partial u}{\partial x_1} = -\dfrac{\partial \Phi_0}{\partial x_2} + \dfrac{\partial \psi}{\partial x_1} \quad \text{in } \Omega \\ u \wedge \dfrac{\partial u}{\partial x_2} = \dfrac{\partial \Phi_0}{\partial x_1} + \dfrac{\partial \psi}{\partial x_2} \quad \text{in } \Omega \; . \end{cases}$$

We have by (11) and (12)

$$(14)\qquad \frac{1}{2}\int_{G_\rho} \mid \nabla u \mid^2 = \frac{1}{2}\int_{G_\rho} \mid \nabla \Phi_0 \mid^2 + \frac{1}{2}\int_{G_\rho} \mid \nabla \psi \mid^2 +$$

$$+\int_{G_\rho} \left[\frac{\partial \psi}{\partial x_1}\left(u_0 \wedge \frac{\partial u_0}{\partial x_1}\right) + \frac{\partial \psi}{\partial x_2}\left(u_0 \wedge \frac{\partial u_0}{\partial x_2}\right)\right] .$$

In Chapter I from [BBH2] it is proved that

$$(15)\qquad \frac{1}{2}\int_{G_\rho} \mid \nabla \Phi_0 \mid^2 = \pi \left(\sum_{j=1}^k d_j^2\right) \log \frac{1}{\rho} + W(a, \overline{d}, g) + O(\rho) \quad \text{as } \rho \to 0 \; .$$

We show now that the third term in the right side of (14) is $O(\rho)$ as $\rho \to 0$. Indeed, since u_0 is an harmonic map and $\psi = 0$ on ∂G, we have

$$(16)\qquad \int_{G_\rho} \left[\frac{\partial \psi}{\partial x_1}\left(u_0 \wedge \frac{\partial u_0}{\partial x_1}\right) + \frac{\partial \psi}{\partial x_2}\left(u_0 \wedge \frac{\partial u_0}{\partial x_2}\right)\right] =$$

$$= \int_{G_\rho} \operatorname{div}\left(\psi(u_0 \wedge \frac{\partial u_0}{\partial x_1}), \psi(u_0 \wedge \frac{\partial u_0}{\partial x_2})\right) = -\sum_{j=1}^k \int_{\partial B(a_j,\rho)} \psi \frac{\partial \Phi_0}{\partial \tau} \; .$$

Around each a_j one may write

$$(17)\qquad \psi = c_j \log \mid x - a_j \mid + \phi_j \; , \qquad \Delta \phi_j = 0$$

$$(18)\qquad \Phi_0 = d_j \log \mid x - a_j \mid + S_j \; , \qquad \Delta S_j = 0 \; .$$

Thus, by (17) and (18),

$$(19)\qquad \int_{\partial B(a_j,\rho)} \psi \frac{\partial \Phi_0}{\partial \tau} = \int_{\partial B(a_j,\rho)} \frac{\partial S_j}{\partial \tau}\left(c_j \log \mid x - a_j \mid + \phi_j\right) =$$

$$= \int_{\partial B(a_j,\rho)} \frac{\partial S_j}{\partial \tau} \phi_j = O(\rho) \quad \text{as } \rho \to 0 \; .$$

All it remains to do now is to estimate $\int_{G_\rho} | \nabla\psi |^2$. We have

$$\int_{G_\rho} | \nabla\psi |^2 = -\sum_{j=1}^{k} \int_{\partial B(a_j,\rho)} \psi \frac{\partial\psi}{\partial\nu} = \tag{20}$$

$$= -\sum_{j=1}^{k} \int_{\partial B(a_j,\rho)} (c_j \log | x - a_j | + \phi_j) \frac{\partial}{\partial\nu}\Big(c_j \log | x - a_j | + \phi_j\Big) =$$

$$= -\sum_{j=1}^{k} \int_{\partial B(a_j,\rho)} (c_j \log\rho + \phi_j)\Big(\frac{c_j}{\rho} + \frac{\partial\phi_j}{\partial\nu}\Big) =$$

$$= 2\pi\Big(\sum_{j=1}^{k} c_j^2\Big) \log\frac{1}{\rho} - 2\pi\sum_{j=1}^{k} c_j\phi_j(a_j) -$$

$$-\Big(\sum_{j=1}^{k} c_j \int_{\partial B(a_j,\rho)} \frac{\partial\phi_j}{\partial\nu}\Big) \log\rho - \sum_{j=1}^{k} c_j \int_{\partial B(a_j,\rho)} \phi_j \frac{\partial\phi_j}{\partial\nu} =$$

$$= 2\pi\Big(\sum_{j=1}^{k} c_j^2\Big) \log\frac{1}{\rho} - 2\pi\sum_{j=1}^{k} c_j\phi_j(a_j) + O(\rho) \quad \text{as } \rho \to 0 \ .$$

So, by (6), (14), (15), (16), (19) and (20) we have obtained

Theorem 2. *For any harmonic map u,*

$$W(u) = W(a, \overline{d}, g) - \pi\sum_{j=1}^{k} c_j\phi_j(a_j) =$$

$$= W(u_0) - \pi\sum_{i\neq j} c_i c_j \log | a_i - a_j | - \pi\sum_{j=1}^{k} c_j\phi(a_j) \ ,$$

where ϕ was defined in (4).

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ASYMPTOTICS FOR THE MINIMIZERS OF THE GINZBURG-LANDAU ENERGY WITH VANISHING WEIGHT

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Abstract. We study the asymptotic behavior of the minimizers for the Ginzburg-Landau energy with a weight which vanishes. We find the link between the growth rate of the weight near its zeroes and the number of singularities of the limiting configuration, as well as their degrees. We give the expression of the corresponding renormalized energy which governs the location of singularities at the limit.

Introduction

F. Bethuel, H. Brezis and F. Hélein have studied in [BBH4] the asymptotic behavior as $\varepsilon \to 0$ of minimizers of the Ginzburg-Landau energy

$$E_\varepsilon(u, G) = E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (1 - |u|^2)^2$$

in the class

$$H^1_g = H^1_g(G) = \{u \in H^1(G; \mathbb{R}^2);\ u = g \text{ on } \partial G\},$$

where $G \subset \mathbb{R}^2$ is a smooth bounded domain and $g : \partial G \to S^1$ is a smooth data with the topological degree $d > 0$.

For each sequence $\varepsilon_n \to 0$, they have proved the existence of a subsequence, also denoted (ε_n) and of a finite configuration $\{a_1, \cdots, a_d\}$ in G such that (u_{ε_n}) converges in certain topologies to $u_\star$, which is the canonical harmonic map with values in S^1 associated to $\{a_1, \cdots, a_d\}$ with degrees $+1$ and to the boundary data g. This means that

$$u_\star(z) = \frac{z - a_1}{|z - a_1|} \cdots \frac{z - a_d}{|z - a_d|}\, e^{i\varphi(z)} \quad \text{in } G \setminus \{a_1, \cdots, a_d\}$$

with

$$
(1) \qquad \begin{cases} \Delta\varphi = 0 & \text{in } G \\ u_\star = g & \text{on } \partial G\,. \end{cases}
$$

Moreover, the configuration $a = (a_1, \cdots, a_d)$ minimizes the renormalized energy $W(a,g)$. The renormalized energy $W(a,\overline{d},g)$ associated to a given configuration $a = (a_1, \cdots, a_k)$ with corresponding degrees $\overline{d} = (d_1, \cdots, d_k)$ and to the boundary data g with $\deg(g, \partial G) = d$, $d = d_1 + \cdots + d_k$ was introduced in [BBH2], [BBH4]. If all d_j equal $+1$ (that is $k = d$) then $W(a,g)$ denotes $W(a,\overline{d},g)$.

In [LR1] we have studied the Ginzburg-Landau energy with weight

$$
E_\varepsilon^w(u) = \frac{1}{2}\int_G \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2}\int_G (1 - \mid u \mid^2)^2 w\ ,
$$

where $w \in C^1(\overline{G})$, $w > 0$ in $\overline{G}$. We proved a similar behavior of minimizers, but the limiting configuration minimizes the modified renormalized energy. More precisely, u_{ε_n} converges to $u_\star$ in certain topologies but now the limiting configuration $a = (a_1, \cdots, a_d)$ is a minimum point of

$$
\widetilde{W}(b,g) = W(b,g) + \frac{\pi}{2}\sum_{j=1}^{d} \log w(b_j)\ , \quad b \in G^d.
$$

A natural question is to see what happens if w vanishes. We first study the case when $w \geq 0$ and it has a unique zero $x_0 \in G$ and suppose that $w(x) \sim\mid x - x_0 \mid^p$ around x_0, where $p > 1$. This means that $w(x) = \mid x - x_0 \mid^p + f(x) \mid x \mid^{p+1}$ in a neighbourhood of x_0, where f is a C^1 function. We show that, up to a subsequence, u_ε converges to a harmonic map $u_\star$ associated to singularities $x_0, a_1, \cdots, a_k$ with $d_0 = \deg(u_\star, x_0) > 0$ and $\deg(u_\star, a_j) = +1$ for $j = 1, \cdots, k$. More precisely, we have (see Theorems 1 and 7)

$$
u_\star(z) = \left(\frac{z - x_0}{\mid z - x_0 \mid}\right)^{d_0} \frac{z - a_1}{\mid z - a_1 \mid} \cdots \frac{z - a_k}{\mid z - a_k \mid}\, e^{i\varphi}
$$

with $d_0 + k = d$. Here φ is such that (1) holds. Remark that in some situations the set $a = (a_1, \cdots, a_k)$ is empty. We next complete this result by finding:

a) the exact value of k as a function of p and d;

b) the position of $a_1, \cdots, a_k$ through the corresponding renormalized energy.

Our main results are the following:

Theorem A. *Assume that* $d < \frac{p}{4} + 1$. *Then* $d_0 = d$ *and* x_0 *is the only singularity of* $u_\star$.

Theorem B. *Assume that* $d \geq \frac{p}{4}+1$ *and that* p *is not an integer multiple of 4. Then* $d_0 = \left[\frac{p}{4}\right]+1$ *(here* $[x]$ *denotes the integer part of the real number* x*).*

Theorem C. *Assume that* $d \geq \frac{p}{4} + 1$ *and that* p *is an integer multiple of 4. Then either* $d_0 = \frac{p}{4}$ *or* $d_0 = \frac{p}{4} + 1$.

Theorem D. *Assume that* $d \geq \frac{p}{4} + 1$ *and* u_ε *converges to the canonical harmonic map associated to the configuration* $a = (x_0, a_1, \cdots, a_k)$ *with degrees* $\overline{d} = (d_0, +1, \cdots, +1)$ *and to the boundary data* g. *Then the limiting configuration* a *minimizes the renormalized energy*

$$\widehat{W}(b) = W(b, \overline{d}, g) + \frac{\pi}{2} \sum_{j=1}^{k} \log w(b_j)$$

among all configurations $b = (x_0, b_1, \cdots, b_k)$.

We show, by considering two examples, that in Theorem C both cases actually occur (see Examples 1 and 3).

The proofs of Theorems A-D follow immediately from Theorems 6, 7, 8 and 9.

1 Estimates of the energy in the case of a ball

We start with a preliminary result.

Theorem 1. *For each sequence* $\varepsilon_n \to 0$, *there exist a subsequence (also denoted by* ε_n*),* k *points* $a_1, \cdots, a_k$ *in* G *and positive integers* $d_0, d_1, \cdots, d_k$ *with* $d_0+d_1+\cdots+d_k = d$ *such that* (u_{ε_n}) *converges in* $H^1_{\mathrm{loc}}(\overline{G}\backslash\{x_0, a_1, \cdots, a_k\}; \mathbb{R}^2)$ *to* $u_\star$, *which is the canonical harmonic map with values in* S^1 *associated to the points* $x_0, a_1, \cdots, a_k$ *with corresponding degrees* $d_0, d_1, \cdots, d_k$ *and to the boundary data* g. *Moreover,* $d_0 \geq 0$ *and* $d_1 = \cdots = d_k = \pm 1$.

Proof. As in [BBH4], the estimate

$$\frac{1}{\varepsilon^2} \int_{G\backslash U} (1- \mid u_\varepsilon \mid^2)^2 w \leq C \tag{2}$$

is fundamental to prove the convergence of (u_ε), where U is an arbitrary neighbourhood of x_0 and $C = C(U)$. The estimate (2) may be obtained with the techniques of Struwe (see [S2]) used by Hong in the case $w > 0$ (see [H]).

Let V be a closed neighbourhood of x_0. With the methods developed in [BBH4], Chapters III-VI, one obtains a finite number of "bad" discs in $G \setminus V$. By this way we find a finite configuration $\{a_1, \cdots, a_k\}$ (k depending on V) in $G \setminus V$ such that, up to a subsequence, (u_{ε_n}) converges in $H^1_{\mathrm{loc}}(\overline{G} \setminus (V \cup \{a_1, \cdots, a_k\}); \mathbb{R}^2)$ to some $u_\star$. The limit $u_\star$ is a harmonic map with values in S^1 and singularities $a_1, \cdots, a_k$, such that the degree of $u_\star$ around each a_j ($j \geq 1$) is some non-zero integer d_j. The fact that all the singularities lie in G follows as in [BBH4], Theorem VI.2.

Taking arbitrary small neighbourhoods V of x_0 and passing to a further subsequence, we obtain by a diagonal argument a sequence (a_k) of points in G without cluster point in $G \setminus \{x_0\}$ and a sequence (d_k) of non-zero integers such that (u_{ε_n}) converges in

$$H^1_{\mathrm{loc}}(\overline{G} \setminus (\{x_0\} \cup \{a_k;\, k \geq 1\}); \mathbb{R}^2)$$

to $u_\star$, which is a harmonic map from $\overline{G} \setminus (\{x_0\} \cup \{a_k;\, k \geq 1\})$ with values in S^1 and singularities a_k of degrees d_k.

As in [BBH4], Theorem III.1,

$$E_\varepsilon(u_\varepsilon) \leq \pi d \log \frac{1}{\varepsilon} + O(1), \quad \text{as } \varepsilon \to 0\ . \tag{3}$$

Taking into account the energy estimates in [BBH4] (see also [LR1]) we obtain that

$$\sum_{j \geq 1} d_j^2 \leq d\,. \tag{4}$$

This means that there is a finite number of singularities a_j, say k.

Denote $d_0 = \deg(u_\star, x_0)$, which is well defined, since x_0 is an isolated singularity. By adapting the proof of Lemma V.2 from [BBH4] in our case and on $G \setminus V$ we obtain that all degrees d_j, $j = 1, \cdots, k$ have the same sign. Moreover, as in Theorem VI.2 from [BBH4], $\mid d_j \mid = +1$, for all $j \geq 1$.

We now prove that $d_0 \geq 0$. Indeed, if not, there would be at least $d + 1$ singularities different from 0. This would contradict (4). ■

We shall see later that $d_0 > 0$ and $d_j = +1$, for all $j = 1, \cdots, k$. This will be done after obtaining stronger energy estimates.

At this stage we are in position to point out the following estimate, which will be used in what follows: for each compact $K \subset \overline{G} \setminus \{x_0, a_1, \cdots, a_k\}$,

$$\|\nabla(u_\varepsilon - u_\star)\|_{L^\infty(K)} \leq C_K \varepsilon \ . \tag{5}$$

This follows with the techniques from [BBH3] in the case of a null degree (see also [M]).

We shall next establish, when G is a ball and $w(x) = |x|^p$, upper and lower bounds for the energy E_ε. These will be accomplished by using the techniques developed in [BBH4], Chapter I. We shall also take into account some results from [LR1] (see Theorem 1).

For fixed $p > 0$, $\varepsilon, R > 0$ and $g(x) = \left(\frac{x}{|x|}\right)^d$, set

$$J_d(\varepsilon, R) = J_d^p(\varepsilon, R) = \min_{H_g^1(B_R)} \left\{ \frac{1}{2} \int_{B_R} |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_{B_R} (1 - |u|^2)^2 |x|^p \right\} .$$

By scaling, it is easy to see that

$$J_d(\varepsilon, R) = J_d\left(\frac{\varepsilon}{R^{1+\frac{p}{2}}}, 1\right) . \tag{6}$$

Hence, in order to obtain an asymptotic formula for J_d^p, it suffices to study the functional $J_d(\varepsilon) := J_d(\varepsilon, 1)$. If $p = 0$, denote $I_d(\varepsilon, R) = J_d^0(\varepsilon, R)$. Throughout, u_ε will denote a point where $J_d(\varepsilon)$ is achieved.

We first establish an upper bound for $J_d(\varepsilon)$.

Theorem 2. *The following estimate holds*

$$J_d(\varepsilon) \leq \frac{2d^2}{p+2} \pi \log \frac{1}{\varepsilon} + O(1), \quad \text{as } \varepsilon \to 0 . \tag{7}$$

Proof. For $\alpha > 0$ and $0 < \varepsilon < 1$, let w_ε be a minimizer of E_ε on $H_g^1(B(0, \varepsilon^\alpha))$. In order to obtain (7), we choose the following comparison function:

$$v_\varepsilon(x) = \begin{cases} \left(\frac{x}{|x|}\right)^d & \text{for } \varepsilon^\alpha \leq |x| \leq 1 \\ w_\varepsilon(x) & \text{for } 0 < |x| < \varepsilon^\alpha . \end{cases}$$

A straightforward computation shows that

$$E_\varepsilon(v_\varepsilon; \{x;\ \varepsilon^\alpha < |x| < 1\}) = \frac{1}{2} \int_{\varepsilon^\alpha < |x| < 1} |\nabla v_\varepsilon|^2 = \pi d^2 \alpha \log \frac{1}{\varepsilon} \quad . \tag{8}$$

On the other hand, using Lemma III.1 in [BBH4] and the fact that $|x|^p \leq \varepsilon^{p\alpha}$ on $B(0,\varepsilon^\alpha)$, we obtain
(9)
$$E_\varepsilon(v_\varepsilon;\ B(0,\varepsilon^\alpha)) \leq I_d(\varepsilon^{1-\frac{p\alpha}{2}},\varepsilon^\alpha) = I_d(\varepsilon^{1-\frac{p+2}{2}\alpha},1) \leq \pi\, d\ \left|\ \log \frac{1}{\varepsilon^{1-\frac{p+2}{2}\alpha}}\ \right| + O(1)\,.$$

Now, choosing $\alpha = \dfrac{2}{p+2}$ and taking into account (8) and (9) we obtain (7). ∎

We next establish a lower bound for the energy.

Theorem 3. *Assume that the only limit point of u_ε obtained in Theorem 1 is $\left(\dfrac{x}{|\,x\,|}\right)^d$, that is 0 is the unique singularity of the limit. Then*

$$J_d(\varepsilon) \geq \frac{2d^2}{p+2}\,\pi\ \log\frac{1}{\varepsilon} - O(1) \quad \textit{as } \varepsilon \to 0\,. \tag{10}$$

Proof. We first estimate $\dfrac{d}{d\varepsilon}\,E_\varepsilon(u_\varepsilon)$ using an idea from [S1]. Let $\varepsilon_1 < \varepsilon_2$. Then
$$E_{\varepsilon_1}(u_{\varepsilon_2}) \geq E_{\varepsilon_1}(u_{\varepsilon_1}) \geq E_{\varepsilon_2}(u_{\varepsilon_1}) \geq E_{\varepsilon_2}(u_{\varepsilon_2})\ .$$
Therefore, if $\nu(\varepsilon) := E_\varepsilon(u_\varepsilon)$ then
$$|\ \nu(\varepsilon_1) - \nu(\varepsilon_2)\ | \leq |\ \varepsilon_1 - \varepsilon_2\ | \cdot \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1^2\varepsilon_2^2}\int_{B_1}(1-|\ u_{\varepsilon_2}\ |^2)^2 w(x)dx\ .$$

This implies that ν is locally Lipschitz on $(0,+\infty)$, that is locally absolutely continuous on $(0,+\infty)$ and ν equals to the integral of its derivative. On the other hand
$$\frac{E_{\varepsilon_1}(u_{\varepsilon_2}) - E_{\varepsilon_2}(u_{\varepsilon_2})}{\varepsilon_1-\varepsilon_2} \leq \frac{E_{\varepsilon_1}(u_{\varepsilon_1}) - E_{\varepsilon_2}(u_{\varepsilon_2})}{\varepsilon_1-\varepsilon_2} \leq \frac{E_{\varepsilon_1}(u_{\varepsilon_1}) - E_{\varepsilon_2}(u_{\varepsilon_1})}{\varepsilon_1-\varepsilon_2}\,.$$
Letting $\varepsilon_1 \nearrow \varepsilon_2$ and $\varepsilon_2 \searrow \varepsilon_1$ we have
$$\nu'(\varepsilon) = \frac{d}{d\varepsilon}E_\varepsilon(u_\varepsilon) = -\frac{1}{2\varepsilon^3}\int_{B_1}(1-|\ u_\varepsilon\ |^2)^2\ |\ x\ |^p \quad \text{a.e. on } (0,+\infty). \tag{11}$$

Recall that u_ε satisfies the equation
$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2}\,u_\varepsilon(1-|\ u_\varepsilon\ |^2)\ |\ x\ |^p & \text{in } B_1 \\ \qquad u_\varepsilon = x^d & \text{on } \partial B_1\,. \end{cases} \tag{12}$$

As in the proof of the Pohozaev identity, multiplying (12) by $(x \cdot \nabla u_\varepsilon)$ and integrating by parts we obtain

$$\int_{\partial B_1} \frac{\partial u_\varepsilon}{\partial \nu}(x\cdot\nabla u_\varepsilon) + \int_{B_1} \sum_{i,j} \frac{\partial u_\varepsilon}{\partial x_j}\left(\delta_{ij}\frac{\partial u_\varepsilon}{\partial x_i} + x_i \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}\right) = \frac{p+2}{4\varepsilon^2}\int_{B_1}(1- \mid u_\varepsilon \mid^2)^2 \mid x \mid^p \ .$$

Therefore

$$(13) \qquad \frac{p+2}{2\varepsilon^2}\int_{B_1}(1- \mid u_\varepsilon \mid^2)^2 \mid x \mid^p = \int_{\partial B_1}\left(\mid \frac{\partial u_\varepsilon}{\partial \tau} \mid^2 - \mid \frac{\partial u_\varepsilon}{\partial \nu} \mid^2\right).$$

Thus

$$(14) \qquad \frac{1}{2\varepsilon^2}\int_{B_1}(1- \mid u_\varepsilon \mid^2)^2 \mid x \mid^p = \frac{2d^2}{p+2}\pi - \frac{1}{p+2}\int_{\partial B_1} \mid \frac{\partial u_\varepsilon}{\partial \nu} \mid^2 \ .$$

Taking into account the estimate (5) we obtain from (14) that

$$(15) \qquad \frac{1}{2\varepsilon^2}\int_{B_1}(1- \mid u_\varepsilon \mid^2)^2 \mid x \mid^p = \frac{2d^2}{p+2}\pi + O(\varepsilon) \quad \text{as } \varepsilon \to 0\,.$$

Integrating (11) from ε to 1 we find together with (15) that

$$(16) \qquad E_\varepsilon(u_\varepsilon) = \frac{2d^2}{p+2}\pi \log\frac{1}{\varepsilon} + O(1) \quad \text{as } \varepsilon \to 0\,.$$

■

Theorem 4. *Suppose, in the case of the ball B_1 and $w(x) = \mid x \mid^p$, that u_{ε_n} converges as in Theorem 1 to $u_\star$ which has singularities 0 with degree d_0 and $a_1, \cdots, a_k$ such that*

$$\deg(u_\star, a_1) = \cdots = \deg(u_\star, a_k) = \pm 1\,.$$

Then

$$(17) \qquad \frac{1}{4\varepsilon_n^2}\int_{B_1}(1- \mid u_{\varepsilon_n} \mid^2)^2 \mid x \mid^p = \frac{d_0^2}{p+2}\pi + \frac{k\pi}{2} + O(\varepsilon) \quad \text{as } n \to \infty\,.$$

Proof. We follow the strategy of the proof of Theorem VII.2 from [BBH4]. From (13) we have that

$$W_n = \frac{1}{4\varepsilon_n^2}(1- \mid u_{\varepsilon_n} \mid^2)^2 \mid x \mid^p$$

is bounded in $L^1(B_1)$ as $n \to \infty$. We also remark at this stage that there exists $C > 0$ such that, for all $\varepsilon > 0$ (and not only for a subsequence),

$$\frac{1}{4\varepsilon^2} \int_{B_1} (1- \mid u_\varepsilon \mid^2)^2 \mid x \mid^p \leq C \, .$$

Indeed, if not, passing to a subsequence ε_n such that (u_{ε_n}) converges, we would contradict the previous result.

By the boundedness of (W_n) it follows its convergence weak $\star$ in $C(\overline{B}_1)^\star$ to a measure $W_\star$ supported by $0, a_1, \cdots, a_k$. Hence

$$W_\star = m_0 \delta_0 + \sum_{j=1}^{k} m_j \delta_{a_j} \quad \text{with } m_j \in \mathbb{R} \, .$$

We now determine m_0.

Consider $B_R = B(0, R)$ for R small enough so that B_R contains no other point a_i $(i \neq 0)$. Multiplying (12) by $x \cdot \nabla u_\varepsilon$ and integrating on B_R we obtain

$$\frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\varepsilon}{\partial \nu} \mid^2 + \frac{p+2}{4\varepsilon^2} \int_{B_R} (1- \mid u_\varepsilon \mid^2)^2 \mid x \mid^p = \tag{18}$$

$$= \frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\varepsilon}{\partial \tau} \mid^2 + \frac{R}{4\varepsilon^2} \int_{\partial B_R} (1- \mid u_\varepsilon \mid^2)^2 \mid x \mid^p \; .$$

Passing to the limit in (18) as $\varepsilon \to 0$ and using the convergence of W_n we find

$$\frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\star}{\partial \nu} \mid^2 + (p+2) m_0 = \frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\star}{\partial \tau} \mid^2 \; . \tag{19}$$

The fact that $u_\star$ is canonical implies that

$$u_\star(x) = \left(\frac{x}{\mid x \mid} \right)^{d_0} e^{i H_0(x)} \quad \text{on } B_R$$

with

$$\Delta H_0 = 0 \, .$$

Therefore, on ∂B_R,

$$\mid \frac{\partial u_\star}{\partial \nu} \mid^2 = \mid d_0 \frac{\partial \theta}{\partial \nu} + \frac{\partial H_0}{\partial \nu} \mid^2 = \mid \frac{\partial H_0}{\partial \nu} \mid^2 \; . \tag{20}$$

$$\mid \frac{\partial u_\star}{\partial \tau} \mid^2 = \mid d_0 \frac{\partial \theta}{\partial \tau} + \frac{\partial H_0}{\partial \tau} \mid^2 = \frac{d_0^2}{R^2} + 2 \frac{d_0}{R} \frac{\partial H_0}{\partial \tau} + \mid \frac{\partial H_0}{\partial \tau} \mid^2 \; . \tag{21}$$

Inserting (20) and (21) into (19) we obtain

$$\frac{R}{2}\int_{\partial B_R} \mid \frac{\partial H_0}{\partial \nu} \mid^2 +(p+2)m_0 = d_0^2\pi + \frac{R}{2}\int_{\partial B_R} \mid \frac{\partial H_0}{\partial \tau} \mid^2 \quad . \tag{22}$$

On the other hand, by multiplying $\Delta H_0 = 0$ with $x \cdot \nabla H_0$ and integrating on B_R we find

$$\frac{R}{2}\int_{\partial B_R} \mid \frac{\partial H_0}{\partial \nu} \mid^2 = \frac{R}{2}\int_{\partial B_R} \mid \frac{\partial H_0}{\partial \tau} \mid^2 \quad . \tag{23}$$

Thus, from (22) and (23) we obtain

$$m_0 = \frac{\pi}{p+2}\, d_0^2 \; .$$

A similar computation for a_j, $j \neq 0$ gives $m_j = \frac{\pi}{2}$ (see [BBH4], Theorem VII.2). ■

Remark 1. By analyzing the proofs of Theorems 3 and 4 we observe that we may replace the weight $\mid x \mid^p$ by a weight which, in a neighbourhood of 0 is of the form $w(x) = \mid x \mid^p + f(x) \mid x \mid^{p+1}$, with $f \in C^1$.

Remark 2. The conclusion of Theorem 4 remains valid for a general domain G and a weight $w(x) = |x|^p$ around 0. In this case, the boundedness of

$$\frac{1}{4\varepsilon^2}\int_G (1- \mid u_\varepsilon \mid^2)^2 w$$

follows by the same computation as in the proof of Theorem 4.

Until now we have obtained a lower bound for the energy under the supplementary hypotheses that $G = B_1$, $g = e^{id\theta}$ and $w(x) = |x|^p$. We now establish a general lower bound for $E_\varepsilon(u_\varepsilon)$ when w is like in Remark 1; this will be useful to deduce the exact value of d_0.

Theorem 5. *Let*

$$C = \liminf_{\varepsilon \to 0} \frac{1}{4\varepsilon^2}\int_G (1- \mid u_\varepsilon \mid^2)^2 w \, . \tag{24}$$

Then

i) $C > 0$.

ii) The following hold:

$$\frac{1}{4\varepsilon^2}\int_G (1- \mid u_\varepsilon \mid^2)^2 w \geq C - O(\varepsilon) \, . \tag{25}$$

and

$$E_\varepsilon(u_\varepsilon) \geq 2C\,\pi\,\log\frac{1}{\varepsilon} - O(\varepsilon)\,. \tag{25'}$$

iii) We have

$$C \geq \min\left\{\frac{(d-\ell)^2}{p+2} + \frac{\ell}{2}\,;\ 0 \leq \ell \leq d\right\}. \tag{26}$$

Proof. ii) Suppose (25) does not hold. Then there are $\varepsilon_n \to 0$ and $C_n \to +\infty$ such that

$$\frac{1}{4\varepsilon_n^2}\int_G (1-\mid u_{\varepsilon_n}\mid^2)^2 w \leq C - C_n\varepsilon_n\,.$$

We may suppose that u_{ε_n} converges as in Theorem 1. Taking into account (18) and the rate of convergence of u_ε away from singularities (see [BBH4], Theorem VI.1) we easily observe that

$$\frac{1}{4\varepsilon_n^2}\int_G (1-\mid u_{\varepsilon_n}\mid^2)^2 w = C + O(\varepsilon_n)\,,$$

which gives a contradiction.

The inequality (25') follows by integrating (11) for small ε.

i),iii) By Theorem 4, any limit point as $\varepsilon \to 0$ of

$$\frac{1}{4\varepsilon^2}\int_G (1-\mid u_\varepsilon\mid^2)^2 w$$

is of the form
$$\frac{(d-\ell)^2}{p+2}\,\pi + \frac{\mid\ell\mid\pi}{2} \qquad \text{with } -d \leq \ell \leq d$$

and i), iii) follow immediately. ∎

Theorem 1'. *Under the assumptions of Theorem 1, we have $d_0 > 0$.*

Proof. We already know that $d_0 \geq 0$. Suppose $d_0 = 0$. Then, as in [LR1], Theorem 1,

$$E_\varepsilon(u_\varepsilon) \geq \pi\,d\,\log\frac{1}{\varepsilon} - C\,.$$

On the other hand, by Theorem 2 and choosing an appropriate test function,

$$E_\varepsilon(u_\varepsilon) \leq \left(\frac{2}{p+2} + (d-1)\right)\pi\,\log\frac{1}{\varepsilon} + C\,.$$

This gives a contradiction. $\blacksquare$

Theorem 6. *Let $G = B_1$, $g(\theta) = e^{id\theta}$ and $w(x) =\mid x \mid^p$. If $d < \dfrac{p}{4} + 1$ then, for the corresponding minimizers u_ε of E_ε, we have*

$$u_\varepsilon(x) \to \left(\frac{x}{\mid x \mid}\right)^d \quad \text{as } \varepsilon \to 0 \,.$$

If p is not an integer multiple of 4 and $d > \dfrac{p}{4} + 1$, then $u_\star$ has singularities $0, a_1, \cdots, a_k$ with degrees $d_0, +1, \cdots, +1$, where $d_0 = \left[\dfrac{p}{4}\right] + 1$.

Proof. We prove the assertion of the theorem by induction. Let $d = 1$ and let k be the number of singularities different from 0. On the one hand, it follows from Theorem 2 that

$$E_\varepsilon(u_\varepsilon) \leq \frac{2\pi}{p+2}\,\log\frac{1}{\varepsilon} + O(1) \quad \text{as } \varepsilon \to 0\,.$$

On the other hand, it follows as in [LR1], Theorem 1 that

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq \pi\, k\, \log\frac{1}{\varepsilon_n} + O(1) \quad \text{as } \varepsilon_n \to 0\,.$$

We thus obtain $k \leq \dfrac{2}{p+2} < 1$, that is $k = 0$.

Suppose now the assertion true for any $0 \leq k \leq d-1$ with $d < \dfrac{p}{4} + 1$. If the conclusion of the theorem does not hold, there is a sequence $\varepsilon_n \to 0$ and there are $k \geq 1$ points $a_1, \cdots, a_k$ in $G \setminus \{0\}$ such that (u_{ε_n}) has at the limit the singularities $a_1, \cdots, a_k$. These singularities have equal degrees $d' = +1$ or $d' = -1$. We shall examine the two cases:

i) If $d' = +1$ then $d_0 < d$. Taking into account the induction hypotheses and Theorem 5 we obtain, for $R > 0$ sufficiently small,

$$E_\varepsilon(u_\varepsilon; B_R) \geq \frac{2d_0^2}{p+2}\,\pi\,\log\frac{1}{\varepsilon} - C\,, \quad \text{as } \varepsilon \to 0\,.$$

Thus

$$E_\varepsilon(u_\varepsilon) \geq \left(\frac{2d_0^2}{p+2} + k\right)\pi\,\log\frac{1}{\varepsilon} - C\,, \quad \text{as } \varepsilon \to 0\,. \tag{27}$$

But Theorem 2 implies

$$E_\varepsilon(u_\varepsilon) \leq \frac{2d^2}{p+2}\,\pi\,\log\frac{1}{\varepsilon} + C\,, \quad \text{as } \varepsilon \to 0\,. \tag{28}$$

If we compare (27) and (28) we find that

$$\frac{2d^2}{p+2} \geq \frac{2d_0^2}{p+2} + k\,.$$

This inequality is clearly false if $k > 0$ and $d_0 > 0$, contradiction.

ii) Let $d' = -1$. There are two cases:

Case 1: $d+k \leq \frac{p}{4}+1$. In this case, the corresponding minimum in (26) for d replaced by $d+k$ is achieved for $\ell = 0$ and we obtain from Theorem 5 that

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq \left(\frac{2(d+k)^2}{p+2} - \delta + k\right)\pi \log\frac{1}{\varepsilon_n} - C \quad \text{as } \varepsilon_n \to 0\,.$$

This contradicts the upper bound (7).

Case 2: $d+k > \frac{p}{4}+1$. In this case, the minimum in (26) (for d replaced by $d+k$) is $> \frac{d^2}{p+2}$. This yields again a contradiction. ■

Theorem 7. *Under the assumptions of Theorem 1, we have* $d_i = +1$, *for* $i = 1, \cdots, k$.
If p *is an integer multiple of 4 and* $d \geq \frac{p}{4}+1$ *then* $d_0 \in \left\{\frac{p}{4}, \frac{p}{4}+1\right\}$.

Proof. The fact that $d_i = +1$ follows as in Theorem 6. The statement that $d_0 \in \left\{\frac{p}{4}, \frac{p}{4}+1\right\}$ for $d \geq \frac{p}{4}+1$ is a consequence of Theorem 5 and of the fact that the quantity

$$\frac{2d_0^2}{p+2} + (d - d_0)$$

atteints its minimum in the set $d_0 \in \{1, \cdots, d\}$ for $d_0 = \frac{p}{4}$ or $d_0 = \frac{p}{4}+1$. ■

2 The renormalized energy

In [BBH4], F. Bethuel, H. Brezis and F. Hélein have introduced the concept of renormalized energy associated to a given configuration of points with prescribed degrees and to a boundary data. They observed that the limiting configuration of singularities is a minimum point of this functional. We shall find the renormalized energy in the case of a ball, say B_1, when the weight is $w(x) = \mid x \mid^p$. In the case of a vanishing weight the introduction of a concept of renormalized energy is useful only for $d \geq \frac{p}{4} + 1$. Indeed, for $d < \frac{p}{4} + 1$ there is only one singularity at the limit, namely the zero of w.

Theorem 8. *Let $g : \partial B_1 \to S^1$, $\deg(g, \partial B_1) = d > \frac{p}{4} + 1$, $w(x) = \mid x \mid^p$. If u_{ε_n} converges to the canonical harmonic map $u_\star$ associated to $a = (0, a_1, \cdots, a_k)$ with corresponding degrees $\overline{d} = (d_0, +1, \cdots, +1)$, then the configuration a minimizes the functional*

$$\widehat{W}(a, g) = W(a, \overline{d}, g) + \frac{\pi}{2} \sum_{j=1}^{k} \log w(a_j) \ .$$

The proof follows the same lines as of the proof of Theorem 1 in [LR1]. ■

It has been observed in the preceding Section that if p is an integer multiple of 4, then $d_0 \in \left\{ \frac{p}{4}, \frac{p}{4} + 1 \right\}$. In what follows we show that both cases may occur.

Example 1. If p is an integer multiple of 4, $G = B_1$, $w(x) = \mid x \mid^p$, $g(\theta) = e^{di\theta}$ and $d = \frac{p}{4} + 1$ then $d_0 = \frac{p}{4} + 1$.
Assume, by contradiction, that $d_0 \neq d$. As observed in Theorem 7, the only possibility in this case is $d_0 = \frac{p}{4}$. By Theorem 8, the limiting configuration $a = (0, a_1)$ with degrees $\overline{d} = (\frac{p}{4}, 1)$ minimizes the functional $\widehat{W}$. We may now make use of the explicit form of the renormalized energy W found in [LR2], Proposition 2:

$$\begin{aligned} W(a, \overline{d}, g) &= -\frac{\pi}{2} p \log \mid a_1 \mid - \pi \log(1 - \mid a_1 \mid^2) - \frac{\pi}{2} p \log(\mid a_1 \mid^2 + 1 - \mid a_1 \mid^2) \\ &= -\frac{\pi}{2} p \log \mid a_1 \mid - \pi \log(1 - \mid a_1 \mid^2) \, . \end{aligned}$$

Hence

$$\widehat{W}(a, g) = -\pi \log(1 - \mid a_1 \mid^2) \, .$$

But this functional does not achieve its infimum on $B_1 \setminus \{0\}$. So, this case is impossible, that is $d_0 = \frac{p}{4} + 1$.

Example 2. If p is an integer multiple of 4, $G = B_1$, $w(x) = \mid x \mid^p$, $g(\theta) = e^{di\theta}$ and $d = \frac{p}{4} + 2$ then $d_0 = \frac{p}{4}$.
Indeed, with the explicit form of the renormalized energy (see [LR2]) we compute $\widehat{W}$ when $d_0 = \frac{p}{4} + 1$ (that is $k = 1$) and $d_0 = \frac{p}{4}$ (that is $k = 2$).

If $d_0 = \dfrac{p}{4} + 1$ then

$$\widehat{W}(0, a_1) = -\pi \,\log\Big(\mid a_1 \mid^2 (1- \mid a_1 \mid^2) \Big)$$

which achieves its infimum on $\overline{B}_1 \setminus \{0\}$ and

$$\inf \widehat{W}(0, a_1) = \pi \,\log 4\,.$$

If $d_0 = \dfrac{p}{4}$ then

$$\widehat{W}(0, a_1, a_2) = -\pi \,\log \mid a_1 - a_2 \mid^2 -\pi \,\log(1- \mid a_1 \mid^2) - \pi \,\log(1- \mid a_2 \mid^2) -$$

$$-\pi \,\log\Big(\mid a_1 - a_2 \mid^2 + (1- \mid a_1 \mid^2)(1- \mid a_2 \mid^2) \Big)\,.$$

In this case, with an argument from [LR2], the infimum of $\widehat{W}(0, a_1, a_2)$ is achieved for $a_1 = -a_2 = 5^{-\frac{1}{4}}$. A straightforward calculation gives

$$\inf \widehat{W}(0, a_1, a_2) < \inf \widehat{W}(0, a_1)$$

which means that $d_0 = \dfrac{p}{4}$.

We next turn to the case of general G, g.

Theorem 9. *Let G be a smooth bounded domain in $\mathbb{R}^2$, $g : \partial G \to S^1$ of topological degree d and $w : \overline{G} \to \mathbb{R}$, $w > 0$ in $\overline{G} \setminus \{x_0\}$, $w(x) = C \mid x - x_0 \mid^p + f(x) \mid x - x_0 \mid^{p+1}$ in a small neighbourhood of x_0, where f is a C^1 function. If $d > \dfrac{p}{4} + 1$ then the limit configuration $a = (0, a_1, \cdots, a_k)$ with degrees $\overline{d} = (d_0, +1, \cdots, +1)$, $d_0 > 0$, minimizes the functional $\widehat{W}(a, g)$.*

The proof is similar as of Theorem 8. ■

We shall now give an example which shows that if p is an integer multiple of 4 and for a general weight w that is like $\mid x \mid^p$ in a neighbourhood of 0, then one can not obtain a general result, in the sense that the zero of the weight might have different degrees at the limit. This example shows that not only the behavior of the weight around its zero is important in the determination of degrees, but also the form of the weight w away from 0.

Example 3. Let $h : [0, 1] \to (0, 1]$ be a C^1 function which equals 1 on $[0, \delta_0]$ and $h(a_1) = \min\limits_{[0,1]} h = \delta > 0$, which will be suitable chosen. We take

$w(x) = h(|\ x\ |)\ |\ x\ |^p$, p an integer multiple of 4 and $g(x) = x^d$ on ∂B_1, where $d = \frac{p}{4} + 1$. We shall choose δ such that

$$W\Big((0),(d)\Big) > W\Big((0,a_1),(d-1,+1)\Big) + \frac{\pi}{2}\ \log(\delta a_1^p)\ .$$

Taking into account Theorems 8 and 9, it follows that this choice of δ gives $d_0 = \frac{p}{4}$.

3 Remarks for the case of a weight with several zeroes

For the sake of simplicity assume w has two zeroes a_1 and a_2 in G and, in small neighbourhoods of a_j,

$$w(x) = |\ x - a_j\ |^{p_j} \qquad \text{with } p_j > 0,\ j = 1, 2\,.$$

We also suppose that each p_j is not an integer multiple of 4. If $d > \left[\frac{p_1}{4}\right] + \left[\frac{p_2}{4}\right] + 2$ it can be proved using the same techniques that u_{ε_n} converges to $u_\star$ which has singularities $a_1, a_2, \cdots, a_k$ of corresponding degrees $d_1 = \left[\frac{p_1}{4}\right] + 1, d_2 = \left[\frac{p_2}{4}\right] + 1, d_3 = \cdots = d_k = +1$. Moreover, the configuration $a = (a_1, a_2, a_3, \cdots, a_k)$ with $\overline{d} = (d_1, d_2, +1, \cdots, +1)$ minimizes the renormalized energy

$$\widehat{W}(a, \overline{d}, g) = W(a, \overline{d}, g) + \frac{\pi}{2} \sum_{j=3}^{k} \log\, w(a_j)\,.$$

The case $d \le \left[\frac{p_1}{4}\right] + \left[\frac{p_2}{4}\right] + 2$ yields a delicate discussion. For example, if $d = 1$, then there is only one singularity at the limit. This is a_1 if

$$\frac{2}{p_1 + 2} < \frac{2}{p_2 + 2}\,, \qquad \text{that is} \qquad p_1 > p_2\,.$$

The case $p_1 = p_2$ is more difficult. If

$$W(a_1, 1, g) < W(a_2, 1, g) \tag{29}$$

then the singularity at the limit is a_1. We cannot conclude when equality holds in (29).

Suppose now $d = 2$ and $p_1 > p_2$. If

$$\frac{8}{p_1+2} < \frac{2}{p_1+2} + \frac{2}{p_2+2} \tag{30}$$

then, at the limit, there is one singularity, namely a_1, of degree $+2$. If

$$\frac{8}{p_1+2} > \frac{2}{p_1+2} + \frac{2}{p_2+2}$$

then there are two singularities at the limit, namely a_1 and a_2 of corresponding degrees $+1$. If the equality holds in (30) we argue in terms of renormalized energy as above.

The discussion may be similarly continued for greater values of d.

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ASYMPTOTICS FOR THE MINIMIZERS OF THE GINZBURG-LANDAU ENERGY WITH VANISHING WEIGHT

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Abstract. We study the asymptotic behavior of the minimizers for the Ginzburg-Landau energy with a weight which vanishes. We find the link between the growth rate of the weight near its zeroes and the number of singularities of the limiting configuration, as well as their degrees. We give the expression of the corresponding renormalized energy which governs the location of singularities at the limit.

Introduction

F. Bethuel, H. Brezis and F. Hélein have studied in [BBH4] the asymptotic behavior as $\varepsilon \to 0$ of minimizers of the Ginzburg-Landau energy

$$E_\varepsilon(u,G) = E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (1-|u|^2)^2$$

in the class

$$H^1_g = H^1_g(G) = \{u \in H^1(G;\mathbb{R}^2);\ u = g \text{ on } \partial G\},$$

where $G \subset \mathbb{R}^2$ is a smooth bounded domain and $g : \partial G \to S^1$ is a smooth data with the topological degree $d > 0$.

For each sequence $\varepsilon_n \to 0$, they have proved the existence of a subsequence, also denoted (ε_n) and of a finite configuration $\{a_1, \cdots, a_d\}$ in G such that (u_{ε_n}) converges in certain topologies to $u_\star$, which is the canonical harmonic map with values in S^1 associated to $\{a_1, \cdots, a_d\}$ with degrees $+1$ and to the boundary data g. This means that

$$u_\star(z) = \frac{z-a_1}{|z-a_1|}\cdots\frac{z-a_d}{|z-a_d|}\, e^{i\varphi(z)} \quad \text{in } G \setminus \{a_1, \cdots, a_d\}$$

with

$$
(1) \qquad \begin{cases} \Delta\varphi = 0 & \text{in } G \\ u_\star = g & \text{on } \partial G\,. \end{cases}
$$

Moreover, the configuration $a = (a_1, \cdots, a_d)$ minimizes the renormalized energy $W(a, g)$. The renormalized energy $W(a, \overline{d}, g)$ associated to a given configuration $a = (a_1, \cdots, a_k)$ with corresponding degrees $\overline{d} = (d_1, \cdots, d_k)$ and to the boundary data g with $\deg(g, \partial G) = d$, $d = d_1 + \cdots + d_k$ was introduced in [BBH2], [BBH4]. If all d_j equal $+1$ (that is $k = d$) then $W(a, g)$ denotes $W(a, \overline{d}, g)$.

In [LR1] we have studied the Ginzburg-Landau energy with weight

$$
E_\varepsilon^w(u) = \frac{1}{2}\int_G \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2}\int_G (1- \mid u \mid^2)^2 w\ ,
$$

where $w \in C^1(\overline{G})$, $w > 0$ in $\overline{G}$. We proved a similar behavior of minimizers, but the limiting configuration minimizes the modified renormalized energy. More precisely, u_{ε_n} converges to $u_\star$ in certain topologies but now the limiting configuration $a = (a_1, \cdots, a_d)$ is a minimum point of

$$
\widetilde{W}(b, g) = W(b, g) + \frac{\pi}{2}\sum_{j=1}^{d} \log w(b_j)\ , \quad b \in G^d.
$$

A natural question is to see what happens if w vanishes. We first study the case when $w \geq 0$ and it has a unique zero $x_0 \in G$ and suppose that $w(x) \sim \mid x - x_0 \mid^p$ around x_0, where $p > 1$. This means that $w(x) = \mid x - x_0 \mid^p + f(x) \mid x \mid^{p+1}$ in a neighbourhood of x_0, where f is a C^1 function. We show that, up to a subsequence, u_ε converges to a harmonic map $u_\star$ associated to singularities $x_0, a_1, \cdots, a_k$ with $d_0 = \deg(u_\star, x_0) > 0$ and $\deg(u_\star, a_j) = +1$ for $j = 1, \cdots, k$. More precisely, we have (see Theorems 1 and 7)

$$
u_\star(z) = \left(\frac{z - x_0}{\mid z - x_0 \mid}\right)^{d_0} \frac{z - a_1}{\mid z - a_1 \mid} \cdots \frac{z - a_k}{\mid z - a_k \mid}\, e^{i\varphi}
$$

with $d_0 + k = d$. Here φ is such that (1) holds. Remark that in some situations the set $a = (a_1, \cdots, a_k)$ is empty. We next complete this result by finding:

a) the exact value of k as a function of p and d;

b) the position of $a_1, \cdots, a_k$ through the corresponding renormalized energy.

Our main results are the following:

Theorem A. *Assume that* $d < \frac{p}{4} + 1$. *Then* $d_0 = d$ *and* x_0 *is the only singularity of* $u_\star$.

Theorem B. *Assume that* $d \geq \frac{p}{4}+1$ *and that* p *is not an integer multiple of 4. Then* $d_0 = \left[\frac{p}{4}\right]+1$ *(here* $[x]$ *denotes the integer part of the real number* x*).*

Theorem C. *Assume that* $d \geq \frac{p}{4} + 1$ *and that* p *is an integer multiple of 4. Then either* $d_0 = \frac{p}{4}$ *or* $d_0 = \frac{p}{4} + 1$.

Theorem D. *Assume that* $d \geq \frac{p}{4} + 1$ *and* u_ε *converges to the canonical harmonic map associated to the configuration* $a = (x_0, a_1, \cdots, a_k)$ *with degrees* $\overline{d} = (d_0, +1, \cdots, +1)$ *and to the boundary data* g. *Then the limiting configuration* a *minimizes the renormalized energy*

$$\widehat{W}(b) = W(b, \overline{d}, g) + \frac{\pi}{2} \sum_{j=1}^{k} \log w(b_j)$$

among all configurations $b = (x_0, b_1, \cdots, b_k)$.

We show, by considering two examples, that in Theorem C both cases actually occur (see Examples 1 and 3).

The proofs of Theorems A-D follow immediately from Theorems 6, 7, 8 and 9.

1 Estimates of the energy in the case of a ball

We start with a preliminary result.

Theorem 1. *For each sequence* $\varepsilon_n \to 0$, *there exist a subsequence (also denoted by* ε_n*),* k *points* $a_1, \cdots, a_k$ *in* G *and positive integers* $d_0, d_1, \cdots, d_k$ *with* $d_0+d_1+\cdots+d_k = d$ *such that* (u_{ε_n}) *converges in* $H^1_{\text{loc}}(\overline{G}\backslash\{x_0, a_1, \cdots, a_k\}; \mathbb{R}^2)$ *to* $u_\star$, *which is the canonical harmonic map with values in* S^1 *associated to the points* $x_0, a_1, \cdots, a_k$ *with corresponding degrees* $d_0, d_1, \cdots, d_k$ *and to the boundary data* g. *Moreover,* $d_0 \geq 0$ *and* $d_1 = \cdots = d_k = \pm 1$.

Proof. As in [BBH4], the estimate

$$\frac{1}{\varepsilon^2} \int_{G\backslash U} (1- \mid u_\varepsilon \mid^2)^2 w \leq C \tag{2}$$

is fundamental to prove the convergence of (u_ε), where U is an arbitrary neighbourhood of x_0 and $C = C(U)$. The estimate (2) may be obtained with the techniques of Struwe (see [S2]) used by Hong in the case $w > 0$ (see [H]).

Let V be a closed neighbourhood of x_0. With the methods developed in [BBH4], Chapters III-VI, one obtains a finite number of "bad" discs in $G \setminus V$. By this way we find a finite configuration $\{a_1, \cdots, a_k\}$ (k depending on V) in $G \setminus V$ such that, up to a subsequence, (u_{ε_n}) converges in $H^1_{\mathrm{loc}}(\overline{G} \setminus (V \cup \{a_1, \cdots, a_k\}); \mathbb{R}^2)$ to some $u_\star$. The limit $u_\star$ is a harmonic map with values in S^1 and singularities $a_1, \cdots, a_k$, such that the degree of $u_\star$ around each a_j ($j \geq 1$) is some non-zero integer d_j. The fact that all the singularities lie in G follows as in [BBH4], Theorem VI.2.

Taking arbitrary small neighbourhoods V of x_0 and passing to a further subsequence, we obtain by a diagonal argument a sequence (a_k) of points in G without cluster point in $G \setminus \{x_0\}$ and a sequence (d_k) of non-zero integers such that (u_{ε_n}) converges in

$$H^1_{\mathrm{loc}}(\overline{G} \setminus (\{x_0\} \cup \{a_k;\ k \geq 1\}); \mathbb{R}^2)$$

to $u_\star$, which is a harmonic map from $\overline{G} \setminus (\{x_0\} \cup \{a_k;\ k \geq 1\})$ with values in S^1 and singularities a_k of degrees d_k.

As in [BBH4], Theorem III.1,

$$E_\varepsilon(u_\varepsilon) \leq \pi d \log \frac{1}{\varepsilon} + O(1), \quad \text{as } \varepsilon \to 0\ . \tag{3}$$

Taking into account the energy estimates in [BBH4] (see also [LR1]) we obtain that

$$\sum_{j \geq 1} d_j^2 \leq d\,. \tag{4}$$

This means that there is a finite number of singularities a_j, say k.

Denote $d_0 = \deg(u_\star, x_0)$, which is well defined, since x_0 is an isolated singularity. By adapting the proof of Lemma V.2 from [BBH4] in our case and on $G \setminus V$ we obtain that all degrees d_j, $j = 1, \cdots, k$ have the same sign. Moreover, as in Theorem VI.2 from [BBH4], $\mid d_j \mid = +1$, for all $j \geq 1$.

We now prove that $d_0 \geq 0$. Indeed, if not, there would be at least $d + 1$ singularities different from 0. This would contradict (4). ∎

We shall see later that $d_0 > 0$ and $d_j = +1$, for all $j = 1, \cdots, k$. This will be done after obtaining stronger energy estimates.

At this stage we are in position to point out the following estimate, which will be used in what follows: for each compact $K \subset \overline{G} \setminus \{x_0, a_1, \cdots, a_k\}$,

$$\|\nabla(u_\varepsilon - u_\star)\|_{L^\infty(K)} \leq C_K \varepsilon \ . \tag{5}$$

This follows with the techniques from [BBH3] in the case of a null degree (see also [M]).

We shall next establish, when G is a ball and $w(x) = |x|^p$, upper and lower bounds for the energy E_ε. These will be accomplished by using the techniques developed in [BBH4], Chapter I. We shall also take into account some results from [LR1] (see Theorem 1).

For fixed $p > 0$, $\varepsilon, R > 0$ and $g(x) = \left(\frac{x}{|x|}\right)^d$, set

$$J_d(\varepsilon, R) = J_d^p(\varepsilon, R) = \min_{H_g^1(B_R)} \left\{ \frac{1}{2} \int_{B_R} |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_{B_R} (1 - |u|^2)^2 |x|^p \right\} .$$

By scaling, it is easy to see that

$$J_d(\varepsilon, R) = J_d\left(\frac{\varepsilon}{R^{1+\frac{p}{2}}}, 1\right) . \tag{6}$$

Hence, in order to obtain an asymptotic formula for J_d^p, it suffices to study the functional $J_d(\varepsilon) := J_d(\varepsilon, 1)$. If $p = 0$, denote $I_d(\varepsilon, R) = J_d^0(\varepsilon, R)$. Throughout, u_ε will denote a point where $J_d(\varepsilon)$ is achieved.

We first establish an upper bound for $J_d(\varepsilon)$.

Theorem 2. *The following estimate holds*

$$J_d(\varepsilon) \leq \frac{2d^2}{p+2}\, \pi \log \frac{1}{\varepsilon} + O(1), \quad \text{as } \varepsilon \to 0 . \tag{7}$$

Proof. For $\alpha > 0$ and $0 < \varepsilon < 1$, let w_ε be a minimizer of E_ε on $H_g^1(B(0, \varepsilon^\alpha))$. In order to obtain (7), we choose the following comparison function:

$$v_\varepsilon(x) = \begin{cases} \left(\frac{x}{|x|}\right)^d & \text{for } \varepsilon^\alpha \leq |x| \leq 1 \\ w_\varepsilon(x) & \text{for } 0 < |x| < \varepsilon^\alpha . \end{cases}$$

A straightforward computation shows that

$$E_\varepsilon(v_\varepsilon; \{x;\ \varepsilon^\alpha < |x| < 1\}) = \frac{1}{2} \int_{\varepsilon^\alpha < |x| < 1} |\nabla v_\varepsilon|^2 = \pi d^2 \alpha \log \frac{1}{\varepsilon} \quad . \tag{8}$$

On the other hand, using Lemma III.1 in [BBH4] and the fact that $|x|^p \leq \varepsilon^{p\alpha}$ on $B(0,\varepsilon^\alpha)$, we obtain

(9)

$$E_\varepsilon(v_\varepsilon;\ B(0,\varepsilon^\alpha)) \leq I_d(\varepsilon^{1-\frac{p\alpha}{2}},\varepsilon^\alpha) = I_d(\varepsilon^{1-\frac{p+2}{2}\alpha},1) \leq \pi\, d\ \mid \log \frac{1}{\varepsilon^{1-\frac{p+2}{2}\alpha}} \mid + O(1)\,.$$

Now, choosing $\alpha = \dfrac{2}{p+2}$ and taking into account (8) and (9) we obtain (7). ■

We next establish a lower bound for the energy.

Theorem 3. *Assume that the only limit point of u_ε obtained in Theorem 1 is $\left(\dfrac{x}{\mid x\mid}\right)^d$, that is 0 is the unique singularity of the limit. Then*

$$J_d(\varepsilon) \geq \frac{2d^2}{p+2}\,\pi\ \log\frac{1}{\varepsilon} - O(1) \quad \text{as } \varepsilon \to 0\,. \tag{10}$$

Proof. We first estimate $\dfrac{d}{d\varepsilon}\,E_\varepsilon(u_\varepsilon)$ using an idea from [S1]. Let $\varepsilon_1 < \varepsilon_2$. Then

$$E_{\varepsilon_1}(u_{\varepsilon_2}) \geq E_{\varepsilon_1}(u_{\varepsilon_1}) \geq E_{\varepsilon_2}(u_{\varepsilon_1}) \geq E_{\varepsilon_2}(u_{\varepsilon_2})\ .$$

Therefore, if $\nu(\varepsilon) := E_\varepsilon(u_\varepsilon)$ then

$$\mid \nu(\varepsilon_1) - \nu(\varepsilon_2) \mid \leq \mid \varepsilon_1 - \varepsilon_2 \mid \cdot \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1^2\varepsilon_2^2}\int_{B_1}(1-\mid u_{\varepsilon_2}\mid^2)^2 w(x)dx\ .$$

This implies that ν is locally Lipschitz on $(0,+\infty)$, that is locally absolutely continuous on $(0,+\infty)$ and ν equals to the integral of its derivative. On the other hand

$$\frac{E_{\varepsilon_1}(u_{\varepsilon_2}) - E_{\varepsilon_2}(u_{\varepsilon_2})}{\varepsilon_1-\varepsilon_2} \leq \frac{E_{\varepsilon_1}(u_{\varepsilon_1}) - E_{\varepsilon_2}(u_{\varepsilon_2})}{\varepsilon_1-\varepsilon_2} \leq \frac{E_{\varepsilon_1}(u_{\varepsilon_1}) - E_{\varepsilon_2}(u_{\varepsilon_1})}{\varepsilon_1-\varepsilon_2}\,.$$

Letting $\varepsilon_1 \nearrow \varepsilon_2$ and $\varepsilon_2 \searrow \varepsilon_1$ we have

$$\nu'(\varepsilon) = \frac{d}{d\varepsilon}E_\varepsilon(u_\varepsilon) = -\frac{1}{2\varepsilon^3}\int_{B_1}(1-\mid u_\varepsilon\mid^2)^2 \mid x\mid^p \quad \text{a.e. on } (0,+\infty). \tag{11}$$

Recall that u_ε satisfies the equation

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2}\,u_\varepsilon(1-\mid u_\varepsilon\mid^2)\mid x\mid^p & \text{in } B_1 \\ u_\varepsilon = x^d & \text{on } \partial B_1\,. \end{cases} \tag{12}$$

As in the proof of the Pohozaev identity, multiplying (12) by $(x \cdot \nabla u_\varepsilon)$ and integrating by parts we obtain

$$\int_{\partial B_1} \frac{\partial u_\varepsilon}{\partial \nu}\,(x\cdot\nabla u_\varepsilon)+\int_{B_1}\sum_{i,j}\frac{\partial u_\varepsilon}{\partial x_j}\left(\delta_{ij}\frac{\partial u_\varepsilon}{\partial x_i}+x_i\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\right)=\frac{p+2}{4\varepsilon^2}\int_{B_1}(1-\mid u_\varepsilon\mid^2)^2\mid x\mid^p\ .$$

Therefore

$$\frac{p+2}{2\varepsilon^2}\int_{B_1}(1-\mid u_\varepsilon\mid^2)^2\mid x\mid^p=\int_{\partial B_1}\left(\mid\frac{\partial u_\varepsilon}{\partial \tau}\mid^2-\mid\frac{\partial u_\varepsilon}{\partial \nu}\mid^2\right). \tag{13}$$

Thus

$$\frac{1}{2\varepsilon^2}\int_{B_1}(1-\mid u_\varepsilon\mid^2)^2\mid x\mid^p=\frac{2d^2}{p+2}\,\pi-\frac{1}{p+2}\int_{\partial B_1}\mid\frac{\partial u_\varepsilon}{\partial \nu}\mid^2\ . \tag{14}$$

Taking into account the estimate (5) we obtain from (14) that

$$\frac{1}{2\varepsilon^2}\int_{B_1}(1-\mid u_\varepsilon\mid^2)^2\mid x\mid^p=\frac{2d^2}{p+2}\,\pi+O(\varepsilon)\quad\text{as } \varepsilon\to 0\,. \tag{15}$$

Integrating (11) from ε to 1 we find together with (15) that

$$E_\varepsilon(u_\varepsilon)=\frac{2d^2}{p+2}\,\pi\,\log\frac{1}{\varepsilon}+O(1)\quad\text{as } \varepsilon\to 0\,. \tag{16}$$

■

Theorem 4. *Suppose, in the case of the ball B_1 and $w(x)=\mid x\mid^p$, that u_{ε_n} converges as in Theorem 1 to $u_\star$ which has singularities 0 with degree d_0 and $a_1,\cdots,a_k$ such that*

$$\deg\,(u_\star,a_1)=\cdots=\deg\,(u_\star,a_k)=\pm 1\,.$$

Then

$$\frac{1}{4\varepsilon_n^2}\int_{B_1}(1-\mid u_{\varepsilon_n}\mid^2)^2\mid x\mid^p=\frac{d_0^2}{p+2}\,\pi+\frac{k\pi}{2}+O(\varepsilon)\quad\textit{as } n\to\infty\,. \tag{17}$$

Proof. We follow the strategy of the proof of Theorem VII.2 from [BBH4]. From (13) we have that

$$W_n=\frac{1}{4\varepsilon_n^2}(1-\mid u_{\varepsilon_n}\mid^2)^2\mid x\mid^p$$

is bounded in $L^1(B_1)$ as $n \to \infty$. We also remark at this stage that there exists $C > 0$ such that, for all $\varepsilon > 0$ (and not only for a subsequence),

$$\frac{1}{4\varepsilon^2} \int_{B_1} (1- \mid u_\varepsilon \mid^2)^2 \mid x \mid^p \leq C\,.$$

Indeed, if not, passing to a subsequence ε_n such that (u_{ε_n}) converges, we would contradict the previous result.

By the boundedness of (W_n) it follows its convergence weak $\star$ in $C(\overline{B}_1)^\star$ to a measure $W_\star$ supported by $0, a_1, \cdots, a_k$. Hence

$$W_\star = m_0 \delta_0 + \sum_{j=1}^{k} m_j \delta_{a_j} \quad \text{with } m_j \in \mathbb{R}\,.$$

We now determine m_0.

Consider $B_R = B(0,R)$ for R small enough so that B_R contains no other point a_i $(i \neq 0)$. Multiplying (12) by $x \cdot \nabla u_\varepsilon$ and integrating on B_R we obtain

$$\begin{aligned}(18)\qquad & \frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\varepsilon}{\partial \nu} \mid^2 + \frac{p+2}{4\varepsilon^2} \int_{B_R} (1- \mid u_\varepsilon \mid^2)^2 \mid x \mid^p = \\ & = \frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\varepsilon}{\partial \tau} \mid^2 + \frac{R}{4\varepsilon^2} \int_{\partial B_R} (1- \mid u_\varepsilon \mid^2)^2 \mid x \mid^p \;.\end{aligned}$$

Passing to the limit in (18) as $\varepsilon \to 0$ and using the convergence of W_n we find

$$(19)\qquad \frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\star}{\partial \nu} \mid^2 + (p+2) m_0 = \frac{R}{2} \int_{\partial B_R} \mid \frac{\partial u_\star}{\partial \tau} \mid^2 \;.$$

The fact that $u_\star$ is canonical implies that

$$u_\star(x) = \left(\frac{x}{\mid x \mid} \right)^{d_0} e^{iH_0(x)} \quad \text{on } B_R$$

with

$$\Delta H_0 = 0\,.$$

Therefore, on ∂B_R,

$$(20)\qquad \mid \frac{\partial u_\star}{\partial \nu} \mid^2 = \mid d_0 \frac{\partial \theta}{\partial \nu} + \frac{\partial H_0}{\partial \nu} \mid^2 = \mid \frac{\partial H_0}{\partial \nu} \mid^2 \;.$$

$$(21)\qquad \mid \frac{\partial u_\star}{\partial \tau} \mid^2 = \mid d_0 \frac{\partial \theta}{\partial \tau} + \frac{\partial H_0}{\partial \tau} \mid^2 = \frac{d_0^2}{R^2} + 2 \frac{d_0}{R} \frac{\partial H_0}{\partial \tau} + \mid \frac{\partial H_0}{\partial \tau} \mid^2 \;.$$

Inserting (20) and (21) into (19) we obtain

$$\frac{R}{2}\int_{\partial B_R} \mid \frac{\partial H_0}{\partial \nu} \mid^2 +(p+2)m_0 = d_0^2\pi + \frac{R}{2}\int_{\partial B_R} \mid \frac{\partial H_0}{\partial \tau} \mid^2 \quad . \tag{22}$$

On the other hand, by multiplying $\Delta H_0 = 0$ with $x\cdot\nabla H_0$ and integrating on B_R we find

$$\frac{R}{2}\int_{\partial B_R} \mid \frac{\partial H_0}{\partial \nu} \mid^2 = \frac{R}{2}\int_{\partial B_R} \mid \frac{\partial H_0}{\partial \tau} \mid^2 \quad . \tag{23}$$

Thus, from (22) and (23) we obtain

$$m_0 = \frac{\pi}{p+2}\, d_0^2 \ .$$

A similar computation for a_j, $j \neq 0$ gives $m_j = \dfrac{\pi}{2}$ (see [BBH4], Theorem VII.2). ■

Remark 1. By analyzing the proofs of Theorems 3 and 4 we observe that we may replace the weight $\mid x \mid^p$ by a weight which, in a neighbourhood of 0 is of the form $w(x) = \mid x \mid^p + f(x) \mid x \mid^{p+1}$, with $f \in C^1$.

Remark 2. The conclusion of Theorem 4 remains valid for a general domain G and a weight $w(x) = |x|^p$ around 0. In this case, the boundedness of

$$\frac{1}{4\varepsilon^2}\int_G (1- \mid u_\varepsilon \mid^2)^2 w$$

follows by the same computation as in the proof of Theorem 4.

Until now we have obtained a lower bound for the energy under the supplementary hypotheses that $G = B_1$, $g = e^{id\theta}$ and $w(x) = |x|^p$. We now establish a general lower bound for $E_\varepsilon(u_\varepsilon)$ when w is like in Remark 1; this will be useful to deduce the exact value of d_0.

Theorem 5. *Let*

$$C = \liminf_{\varepsilon \to 0} \frac{1}{4\varepsilon^2}\int_G (1- \mid u_\varepsilon \mid^2)^2 w \, . \tag{24}$$

Then

i) $C > 0$.

ii) The following hold:

$$\frac{1}{4\varepsilon^2}\int_G (1- \mid u_\varepsilon \mid^2)^2 w \geq C - O(\varepsilon) \, . \tag{25}$$

and

$$(25') \qquad E_\varepsilon(u_\varepsilon) \geq 2C\,\pi\,\log\frac{1}{\varepsilon} - O(\varepsilon)\,.$$

iii) We have

$$(26) \qquad C \geq \min\left\{\frac{(d-\ell)^2}{p+2} + \frac{\ell}{2}\,;\ 0 \leq \ell \leq d\right\}.$$

Proof. ii) Suppose (25) does not hold. Then there are $\varepsilon_n \to 0$ and $C_n \to +\infty$ such that

$$\frac{1}{4\varepsilon_n^2}\int_G (1-\mid u_{\varepsilon_n}\mid^2)^2 w \leq C - C_n\varepsilon_n\,.$$

We may suppose that u_{ε_n} converges as in Theorem 1. Taking into account (18) and the rate of convergence of u_ε away from singularities (see [BBH4], Theorem VI.1) we easily observe that

$$\frac{1}{4\varepsilon_n^2}\int_G (1-\mid u_{\varepsilon_n}\mid^2)^2 w = C + O(\varepsilon_n)\,,$$

which gives a contradiction.

The inequality (25') follows by integrating (11) for small ε.

i),iii) By Theorem 4, any limit point as $\varepsilon \to 0$ of

$$\frac{1}{4\varepsilon^2}\int_G (1-\mid u_\varepsilon\mid^2)^2 w$$

is of the form
$$\frac{(d-\ell)^2}{p+2}\,\pi + \frac{\mid \ell \mid \pi}{2} \qquad \text{with } -d \leq \ell \leq d$$

and i), iii) follow immediately. ■

Theorem 1'. *Under the assumptions of Theorem 1, we have* $d_0 > 0$.

Proof. We already know that $d_0 \geq 0$. Suppose $d_0 = 0$. Then, as in [LR1], Theorem 1,

$$E_\varepsilon(u_\varepsilon) \geq \pi\,d\,\log\frac{1}{\varepsilon} - C\,.$$

On the other hand, by Theorem 2 and choosing an appropriate test function,

$$E_\varepsilon(u_\varepsilon) \leq \left(\frac{2}{p+2} + (d-1)\right)\pi\,\log\frac{1}{\varepsilon} + C\,.$$

This gives a contradiction. ■

Theorem 6. *Let $G = B_1$, $g(\theta) = e^{id\theta}$ and $w(x) = | x |^p$. If $d < \dfrac{p}{4} + 1$ then, for the corresponding minimizers u_ε of E_ε, we have*

$$u_\varepsilon(x) \to \left(\frac{x}{| x |} \right)^d \quad \text{as } \varepsilon \to 0 \,.$$

If p is not an integer multiple of 4 and $d > \dfrac{p}{4} + 1$, then $u_\star$ has singularities $0, a_1, \cdots, a_k$ with degrees $d_0, +1, \cdots, +1$, where $d_0 = \left[\dfrac{p}{4} \right] + 1$.

Proof. We prove the assertion of the theorem by induction. Let $d = 1$ and let k be the number of singularities different from 0. On the one hand, it follows from Theorem 2 that

$$E_\varepsilon(u_\varepsilon) \leq \frac{2\pi}{p+2} \log \frac{1}{\varepsilon} + O(1) \quad \text{as } \varepsilon \to 0 \,.$$

On the other hand, it follows as in [LR1], Theorem 1 that

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq \pi \, k \log \frac{1}{\varepsilon_n} + O(1) \quad \text{as } \varepsilon_n \to 0 \,.$$

We thus obtain $k \leq \dfrac{2}{p+2} < 1$, that is $k = 0$.

Suppose now the assertion true for any $0 \leq k \leq d - 1$ with $d < \dfrac{p}{4} + 1$. If the conclusion of the theorem does not hold, there is a sequence $\varepsilon_n \to 0$ and there are $k \geq 1$ points $a_1, \cdots, a_k$ in $G \setminus \{0\}$ such that (u_{ε_n}) has at the limit the singularities $a_1, \cdots, a_k$. These singularities have equal degrees $d' = +1$ or $d' = -1$. We shall examine the two cases:

i) If $d' = +1$ then $d_0 < d$. Taking into account the induction hypotheses and Theorem 5 we obtain, for $R > 0$ sufficiently small,

$$E_\varepsilon(u_\varepsilon; B_R) \geq \frac{2d_0^2}{p+2} \pi \log \frac{1}{\varepsilon} - C \,, \quad \text{as } \varepsilon \to 0 \,.$$

Thus

$$E_\varepsilon(u_\varepsilon) \geq \left(\frac{2d_0^2}{p+2} + k \right) \pi \log \frac{1}{\varepsilon} - C \,, \quad \text{as } \varepsilon \to 0 \,. \tag{27}$$

But Theorem 2 implies

$$E_\varepsilon(u_\varepsilon) \leq \frac{2d^2}{p+2} \pi \log \frac{1}{\varepsilon} + C \,, \quad \text{as } \varepsilon \to 0 \,. \tag{28}$$

If we compare (27) and (28) we find that

$$\frac{2d^2}{p+2} \geq \frac{2d_0^2}{p+2} + k\,.$$

This inequality is clearly false if $k > 0$ and $d_0 > 0$, contradiction.

ii) Let $d' = -1$. There are two cases:

Case 1: $d + k \leq \frac{p}{4} + 1$. In this case, the corresponding minimum in (26) for d replaced by $d + k$ is achieved for $\ell = 0$ and we obtain from Theorem 5 that

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq \left(\frac{2(d+k)^2}{p+2} - \delta + k \right) \pi \log \frac{1}{\varepsilon_n} - C \quad \text{as } \varepsilon_n \to 0\,.$$

This contradicts the upper bound (7).

Case 2: $d + k > \frac{p}{4} + 1$. In this case, the minimum in (26) (for d replaced by $d + k$) is $> \frac{d^2}{p+2}$. This yields again a contradiction. ∎

Theorem 7. *Under the assumptions of Theorem 1, we have* $d_i = +1$, *for* $i = 1, \cdots, k$.
If p *is an integer multiple of 4 and* $d \geq \frac{p}{4} + 1$ *then* $d_0 \in \left\{ \frac{p}{4}, \frac{p}{4} + 1 \right\}$.

Proof. The fact that $d_i = +1$ follows as in Theorem 6. The statement that $d_0 \in \left\{ \frac{p}{4}, \frac{p}{4} + 1 \right\}$ for $d \geq \frac{p}{4} + 1$ is a consequence of Theorem 5 and of the fact that the quantity

$$\frac{2d_0^2}{p+2} + (d - d_0)$$

atteints its minimum in the set $d_0 \in \{1, \cdots, d\}$ for $d_0 = \frac{p}{4}$ or $d_0 = \frac{p}{4} + 1$. ∎

2 The renormalized energy

In [BBH4], F. Bethuel, H. Brezis and F. Hélein have introduced the concept of renormalized energy associated to a given configuration of points with prescribed degrees and to a boundary data. They observed that the limiting configuration of singularities is a minimum point of this functional. We shall find the renormalized energy in the case of a ball, say B_1, when the weight is $w(x) = |x|^p$. In the case of a vanishing weight the introduction of a concept of renormalized energy is useful only for $d \geq \frac{p}{4}+1$. Indeed, for $d < \frac{p}{4}+1$ there is only one singularity at the limit, namely the zero of w.

Theorem 8. *Let $g : \partial B_1 \to S^1$, $\deg(g, \partial B_1) = d > \frac{p}{4}+1$, $w(x) = \mid x \mid^p$. If u_{ε_n} converges to the canonical harmonic map $u_\star$ associated to $a = (0, a_1, \cdots, a_k)$ with corresponding degrees $\overline{d} = (d_0, +1, \cdots, +1)$, then the configuration a minimizes the functional*

$$\widehat{W}(a, g) = W(a, \overline{d}, g) + \frac{\pi}{2} \sum_{j=1}^{k} \log w(a_j) \ .$$

The proof follows the same lines as of the proof of Theorem 1 in [LR1]. ■

It has been observed in the preceding Section that if p is an integer multiple of 4, then $d_0 \in \left\{ \frac{p}{4}, \frac{p}{4}+1 \right\}$. In what follows we show that both cases may occur.

Example 1. If p is an integer multiple of 4, $G = B_1$, $w(x) = \mid x \mid^p$, $g(\theta) = e^{di\theta}$ and $d = \frac{p}{4}+1$ then $d_0 = \frac{p}{4}+1$.
Assume, by contradiction, that $d_0 \neq d$. As observed in Theorem 7, the only possibility in this case is $d_0 = \frac{p}{4}$. By Theorem 8, the limiting configuration $a = (0, a_1)$ with degrees $\overline{d} = (\frac{p}{4}, 1)$ minimizes the functional $\widehat{W}$. We may now make use of the explicit form of the renormalized energy W found in [LR2], Proposition 2:

$$\begin{aligned} W(a, \overline{d}, g) &= -\frac{\pi}{2} p \log \mid a_1 \mid - \pi \log(1 - \mid a_1 \mid^2) - \frac{\pi}{2} p \log(\mid a_1 \mid^2 + 1 - \mid a_1 \mid^2) \\ &= -\frac{\pi}{2} p \log \mid a_1 \mid - \pi \log(1 - \mid a_1 \mid^2) \, . \end{aligned}$$

Hence

$$\widehat{W}(a, g) = -\pi \log(1 - \mid a_1 \mid^2) \, .$$

But this functional does not achieve its infimum on $B_1 \setminus \{0\}$. So, this case is impossible, that is $d_0 = \frac{p}{4}+1$.

Example 2. If p is an integer multiple of 4, $G = B_1$, $w(x) = \mid x \mid^p$, $g(\theta) = e^{di\theta}$ and $d = \frac{p}{4}+2$ then $d_0 = \frac{p}{4}$.
Indeed, with the explicit form of the renormalized energy (see [LR2]) we compute $\widehat{W}$ when $d_0 = \frac{p}{4}+1$ (that is $k = 1$) and $d_0 = \frac{p}{4}$ (that is $k = 2$).

If $d_0 = \dfrac{p}{4} + 1$ then

$$\widehat{W}(0, a_1) = -\pi \, \log\Big(| \, a_1 \, |^2 \, (1 - | \, a_1 \, |^2) \Big)$$

which achieves its infimum on $\overline{B}_1 \setminus \{0\}$ and

$$\inf \widehat{W}(0, a_1) = \pi \, \log 4 \, .$$

If $d_0 = \dfrac{p}{4}$ then

$$\widehat{W}(0, a_1, a_2) = -\pi \, \log | \, a_1 - a_2 \, |^2 - \pi \, \log(1 - | \, a_1 \, |^2) - \pi \, \log(1 - | \, a_2 \, |^2) -$$

$$-\pi \, \log\Big(| \, a_1 - a_2 \, |^2 + (1 - | \, a_1 \, |^2)(1 - | \, a_2 \, |^2) \Big) \, .$$

In this case, with an argument from [LR2], the infimum of $\widehat{W}(0, a_1, a_2)$ is achieved for $a_1 = -a_2 = 5^{-\frac{1}{4}}$. A straightforward calculation gives

$$\inf \widehat{W}(0, a_1, a_2) < \inf \widehat{W}(0, a_1)$$

which means that $d_0 = \dfrac{p}{4}$.

We next turn to the case of general G, g.

Theorem 9. *Let G be a smooth bounded domain in $\mathbb{R}^2$, $g : \partial G \to S^1$ of topological degree d and $w : \overline{G} \to \mathbb{R}$, $w > 0$ in $\overline{G} \setminus \{x_0\}$, $w(x) = C \mid x - x_0 \mid^p + f(x) \mid x - x_0 \mid^{p+1}$ in a small neighbourhood of x_0, where f is a C^1 function. If $d > \dfrac{p}{4} + 1$ then the limit configuration $a = (0, a_1, \cdots, a_k)$ with degrees $\overline{d} = (d_0, +1, \cdots, +1)$, $d_0 > 0$, minimizes the functional $\widehat{W}(a, g)$.*

The proof is similar as of Theorem 8. ■

We shall now give an example which shows that if p is an integer multiple of 4 and for a general weight w that is like $| \, x \, |^p$ in a neighbourhood of 0, then one can not obtain a general result, in the sense that the zero of the weight might have different degrees at the limit. This example shows that not only the behavior of the weight around its zero is important in the determination of degrees, but also the form of the weight w away from 0.

Example 3. Let $h : [0, 1] \to (0, 1]$ be a C^1 function which equals 1 on $[0, \delta_0]$ and $h(a_1) = \min\limits_{[0,1]} h = \delta > 0$, which will be suitable chosen. We take

$w(x) = h(\mid x \mid) \mid x \mid^p$, p an integer multiple of 4 and $g(x) = x^d$ on ∂B_1, where $d = \frac{p}{4} + 1$. We shall choose δ such that

$$W\Big((0),(d)\Big) > W\Big((0,a_1),(d-1,+1)\Big) + \frac{\pi}{2}\log(\delta a_1^p)\ .$$

Taking into account Theorems 8 and 9, it follows that this choice of δ gives $d_0 = \frac{p}{4}$.

3 Remarks for the case of a weight with several zeroes

For the sake of simplicity assume w has two zeroes a_1 and a_2 in G and, in small neighbourhoods of a_j,

$$w(x) = \mid x - a_j \mid^{p_j} \qquad \text{with } \ p_j > 0,\ j = 1, 2\,.$$

We also suppose that each p_j is not an integer multiple of 4. If $d > \left[\frac{p_1}{4}\right] + \left[\frac{p_2}{4}\right] + 2$ it can be proved using the same techniques that u_{ε_n} converges to $u_\star$ which has singularities $a_1, a_2, \cdots, a_k$ of corresponding degrees $d_1 = \left[\frac{p_1}{4}\right] + 1, d_2 = \left[\frac{p_2}{4}\right] + 1, d_3 = \cdots = d_k = +1$. Moreover, the configuration $a = (a_1, a_2, a_3, \cdots, a_k)$ with $\overline{d} = (d_1, d_2, +1, \cdots, +1)$ minimizes the renormalized energy

$$\widehat{W}(a, \overline{d}, g) = W(a, \overline{d}, g) + \frac{\pi}{2}\sum_{j=3}^{k} \log\, w(a_j)\,.$$

The case $d \leq \left[\frac{p_1}{4}\right] + \left[\frac{p_2}{4}\right] + 2$ yields a delicate discussion. For example, if $d = 1$, then there is only one singularity at the limit. This is a_1 if

$$\frac{2}{p_1 + 2} < \frac{2}{p_2 + 2}\,, \qquad \text{that is} \qquad p_1 > p_2\,.$$

The case $p_1 = p_2$ is more difficult. If

$$W(a_1, 1, g) < W(a_2, 1, g) \tag{29}$$

then the singularity at the limit is a_1. We cannot conclude when equality holds in (29).

Suppose now $d = 2$ and $p_1 > p_2$. If

$$\frac{8}{p_1+2} < \frac{2}{p_1+2} + \frac{2}{p_2+2} \tag{30}$$

then, at the limit, there is one singularity, namely a_1, of degree $+2$. If

$$\frac{8}{p_1+2} > \frac{2}{p_1+2} + \frac{2}{p_2+2}$$

then there are two singularities at the limit, namely a_1 and a_2 of corresponding degrees $+1$. If the equality holds in (30) we argue in terms of renormalized energy as above.

The discussion may be similarly continued for greater values of d.

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